# A Generalization of Gilmore's Theorem*

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## 1 Helly Families of Order $h$

Let us consider a set $X_0$; a finite family $\mathcal{F} = \{X_j \mid j \in N\} = \{X_1, X_2, \ldots, X_n\}$ of subsets of $X_0$ is a Helly family of order $h$ if, for all $J \subset N$, the condition

$$(\forall I \subset J,\ |I| \leq h) : \quad \bigcap_{i \in I} X_j \neq \emptyset$$

implies

$$\bigcap_{j \in J} X_j \neq \emptyset.$$

A example of a family having this property is supplied by convex compacts of $\mathbb{R}^{h-1}$: that is Helly's theorem. Another example is the family of maximum cliques of the graph $L(K_n^{n-h+1})$, $n > 2h$, (the vertices of which are all subsets with $n - h + 1$ elements of a set with $n$ elements, two vertices being joined iff the corresponding subsets have a non-void intersection); this is a consequence of another famous theorem, that of Erdős, Chao Ko and Radó.

The following theorem which is the essential tool of the results in this article, is another most simple characterization:

**Theorem 1** *$\mathcal{F} = \{X_j \mid j \in N\}$ is a Helly family of order $h$ if and only if for all $A \subset X_0$ such that $|A| = h + 1$,*

$$\bigcap_{|X_j \cap A| \geq h} X_j \neq \emptyset. \tag{1}$$

**Proof:** The Helly property implies the condition of the theorem; indeed, if we consider those $X_i$'s for which $|X_i \cap A| \geq h$, and if we take just $h$ of them, their intersection is non-void (by the pigeonhole principle); hence (1) by Helly's property.

The condition of the theorem implies Helly's property; indeed, let $\{X_1, \ldots, X_n\}$ be a family satisfying the condition (1), and let $J \subset N$ such that

$$(\forall I \subset J,\ |I| \leq h) : \quad \bigcap_{j \in I} X_j \neq \emptyset.$$

We are going to show by induction with respect to $|J|$ that this implies $\bigcap_{j \in J} X_j \neq \emptyset$.

*In M. Fiedler, editor, *Recent Advances in Graph Theory*, Proceedings 2nd Czechoslovak Symposium, pages 49–55, Academia, Prague, 1975.

This is trivially true for $|J| \leq h$; assume thus $|J| > h$. Let $j_1, j_2, \ldots, j_{h+1}$ be different elements of $J$. Then,

$$(\forall I \subset J \setminus \{j_1\},\ |I| \leq h): \quad \bigcap_{j \in I} X_j \neq \emptyset.$$

Consequently, by the induction hypothesis,

$$\bigcap_{j \in J \setminus \{j_1\}} X_j \neq \emptyset.$$

Let $a_1$ be an element of this non-void intersection; similarly, let $a_2, a_3, \ldots, a_{h+1}$ be defined. These elements are all distinct (otherwise the proposition would be proved). Put

$$A = \{a_1, \ldots, a_{h+1}\}.$$

Thus,

$$|X_j \cap A| \geq h \quad (j \in J),$$

whence

$$\bigcap_{j \in J} X_j \neq \emptyset,$$

completing the proof. □

Let us remark that the announced result clearly shows that the convex sets in the plane form a Helly family of order 3: if four points $a$, $b$, $c$, $d$ generate a convex quadrilateral, in this order, the convex sets containing three of these points all contain the point of intersection of the diagonals $ac$ and $bd$; and if they do not generate a convex quadrilateral, all these convex sets contain that of the points which is not extreme of the convex hull of $a$, $b$, $c$, $d$. The same reasoning can be applied to convex sets in $\mathbb{R}^n$.

## 2 Stables and Transversals of a Hypergraph

Let $H$ be a hypergraph with $m$ edges $E_1, \ldots, E_m$, subsets of the set of vertices $X = \bigcup_{i=1}^m E_i$. $H$ is called a *Sperner hypergraph* if $E_i \subset E_j \Longrightarrow i = j$. The *rank* of $H$ is $r(H) = \max |E_i|$.

A set $S \subset X$ is called *stable* if it does not contain any edge; a set $T \subset X$ is called *transversal* it is meets all the edges. Let $\operatorname{St} H$ denote the set of maximal stables, $\operatorname{Tr} H$ the set of minimal transversals. $\operatorname{Tr} H$ is a simple hypergraph. Clearly, $S \in \operatorname{St} H$ iff $X \setminus S \in \operatorname{Tr} H$. Let us mention the fundamental properties:

**Proposition 1** *If $H$ and $H'$ are Sperner hypergraphs then*
*(1)* $\operatorname{Tr}(\operatorname{Tr} H) = H$,
*(2)* $\operatorname{Tr}(H \cup H') = \min\{T \cup T' \mid T \in \operatorname{Tr} H,\ T' \in \operatorname{Tr} H'\}$.

Let $\mathcal{F} = \{X_1, X_2, \ldots, X_n\}$ be a family of subsets of a finite set $E = \{e_i \mid i \in M\}$. A family $\mathcal{E} = \{E_i \mid i \in M\}$ of subsets of $\{x_1, x_2, \ldots, x_n\}$ is called dual of $\mathcal{F}$ if $E_k$ is the set of those $x_j$ for which $X_j \ni e_k$. We write then $\mathcal{E} = \mathcal{F}^*$.

Another way to characterize $\operatorname{St} H$ is to make use of duality. A set $A$ is called *costable* of $H$ if

$$A \not\subset E_i \quad (i = 1, 2, \ldots, m).$$

Let $\mathcal{F}$ be the family of minimal costables of $H$ and put

$$\Phi(H) = \min_{F \in \mathcal{F}} |F|\,,$$
$$\Phi'(H) = \max_{F \in \mathcal{F}} |F|\,.$$

Similarly, put

$$\tau(H) = \max_{T \in \operatorname{Tr} H} |T|\,,$$
$$\tau'(H) = \max_{T \in \operatorname{Tr} H} |T|\,.$$

**Lemma 1** *Let $H = \{E_1, E_2, \ldots, E_m\}$ be a hypergraph on $X$. For $x \in X$ and $I \subset M = \{1, \ldots, m\}$, denote by $d_{H_I}(x)$ the degree of $x$ in the partial hypergraph $H_I = \{E_i \mid i \in I\}$, i.e. $|\{i \mid i \in I,\ E_i \ni x\}|$; let $\bar{H} = \{X \setminus E_i \mid i \in M\}$. The following conditions are equivalent:*
*(1) $H^*$ is a Helly family of order $h$,*
*(2) for all $I \in M$ with $|I| = h+1$ there exists an $E \in H$ containing $\{x \mid x \in X,\ d_{H_I}(x) \geq h\}$,*
*(3) $\Phi'(H) \leq h$,*
*(4) $\tau'(H) \leq h$,*
*(5) for all $I \in M$ with $|I| = h+1$ there exists an $\bar{E} \in \bar{H}$ contained in $\{x \mid x \in X;\ d_{\bar{H}_I} > 1\}$.*

**Proof:** The fact that (1) $\Longleftrightarrow$ (2) follows from Theorem 1.

Let us show that (1) $\Longleftrightarrow$ (3). Let $J \subset M$. Set $A = \{x_j \mid j \in J\}$. The condition

$$(\forall I \subset J,\ |I| \leq h): \quad \bigcap_{j \in I} X_j \neq \emptyset$$

is equivalent with

$$(\forall F \subset A,\ |F| \leq h)(\exists i): \quad E_i \supset F$$

i.e.:

$$A \text{ contains no costables of cardinality } \leq h.$$

On the other hand, the condition

$$\bigcap_{j \in J} X_j \neq \emptyset$$

is equivalent with

$$(\exists i): \quad E_i \supset A$$

or:

$$A \text{ is not a costable.}$$

Thus $H^*$ is a Helly family of order $h$ iff every costable $A$ contains a costable of cardinality $\leq h$, i.e.:

$$\Phi'(H) \leq h.$$

Let us show that (3) $\Longleftrightarrow$ (4). Indeed, $A$ is a costable of $H$ iff $A$ is a transversal of $\bar{H}$. Thus

$$\begin{aligned} \tau'(\bar{H}) &= \Phi'(H), \\ \tau(\bar{H}) &= \Phi(H). \end{aligned}$$

Finally, let us show that (2) $\Longleftrightarrow$ (5). Indeed,

$$X \smallsetminus \{x \mid d_{H_I}(x) \geq h\} = \{x \mid d_{H_I}(x) < h\} = \{x \mid d_{\bar{H}_I}(x) > 1\}.$$

□

The following theorem generalizes a result by P. Gilmore:

**Theorem 2** *Let $(H = (X, \mathcal{E})$ be a Sperner hypergraph. Then $\mathcal{E} = \{E_i \mid i \in M\}$ is the family of maximal stables of a hypergraph in $X$ of rank $\leq h$ iff for each $I \subset M$ such that $|I| = h+1$ there exists an $E \in \mathcal{E}$ containing*

$$\{x \mid d_{H_I}(x) > h\}.$$

**Proof:** (i) Assume $\mathcal{E}$ is the family of maximal stables of $H'$ of rank $\leq h$; let us show that $H^* = \{X_j \mid j \in N\}$, the dual of $H$, is a Helly family of order $h$. Let us consider a set $J \subset N$ and define

$$A = \{x_j \mid j \in J\}.$$

If

$$\forall I \subset J,\ |I| \leq h): \quad \bigcap_{j \in I} X_j \neq \emptyset$$

then

$$(\forall F \subset A,\ |F| \leq h)(\exists E_i \in H): \quad E_i \supset F.$$

Since $E_i$ is a stable of $H'$, it does not contain any edge of $H'$ so that

$$(\forall F \subset A,\ |F| \leq h): \quad F \notin H'.$$

Thus $A$ is a stable of $H'$; let $E_0$ be the maximal stable of $H'$ containing $A$.
Since $x_j \in E_0$ for all $j \in J$, it follows from duality that

$$\bigcap_{j \in J} X_j \neq \emptyset.$$

(ii) Let us show the existence of a hypergraph $H'$ of rank $\leq h$ in $X$ such that $H = \mathrm{St}\, H'$.
Define

$$\mathcal{F} = \{F \mid |F| \leq h,\ F \not\subset E_i \text{ for all } i\}.$$

Since by Lemma 1, $\Phi'(H) \leq h$, it follows that $\mathcal{F} \neq \emptyset$; consequently, $H' = (X, \mathcal{F})$ is a hypergraph in $X$ (with possibly isolated points) of rank $\leq h$.
By definition of $H$, each $E \in H$ is stable for $H'$.
Let $J \subset M$ and $A = \{x_j \mid j \in J\}$. By Lemma 1, each of the conditions implies the next following conditions:

(1) $A$ is stable for $H'$,
(2) $(\forall F \subset A,\ |F| \leq h): \quad F \notin H'$,
(3) $(\forall F \subset A,\ |F| \leq h)(\exists i): \quad F \subset E_i$,
(4) $(\forall I \subset J,\ |I| \leq h): \quad \bigcap_{j \in I} X_j \neq \emptyset$,
(5) $\bigcap_{j \in J} X_j \neq \emptyset$,
(6) $A$ is contained in an $E_i \in H$.

Thus each maximal stable $A$ of $H'$ is contained in an $E_i \in H$; since $E_i$ is stable for $H'$, $E_i = A$ so that

$$\mathrm{St}\, H' \subset H.$$

Since $H$ is Sperner, $\mathrm{St}\, H' = H$ which completes the proof. □

In the case $h = 2$ we obtain the theorem of Gilmore [2]. The following result is another application of Theorem 1:

**Theorem 3** *A Sperner hypergraph $H$ has the property that $\mathrm{Tr}\, H$ is uniform iff for each transversal $T$ and each family $H'$ of $H$ such that $D(H') = \{x \mid d_{H'}(x) > 1\}$ is stable,*

$$\left|H'\right| \leq |T| \,.$$

**Proof:** To prove this, observe that since the conditions (4) and (5) of Lemma 1 are equivalent for $\bar{H}$, the same is true for $H$.

Put $h = \tau(H)$. $\mathrm{Tr}\, H$ is uniform iff

$$\tau'(H) \leq h.$$

Thus, according to (5), if $D(H')$ is stable then

$$\left|H''\right| \leq h = \tau(H).$$

□

**Corollary 1** *If $H$ is an equilibrated Sperner hypergraph which does not admit a perfect matching then $\mathrm{Tr}\, H$ is not uniform.*

Indeed, if $H = (X, \mathcal{E})$ is equilibrated then as is known there exists a transversal $T \subset X$ and a maximal matching $\mathcal{F} \subset \mathcal{E}$ such that $|T| = |\mathcal{F}|$. Let $E_0$ be an edge of $H$ containing a point of $H$ not saturated in $\mathcal{F}$. Let $H' = \mathcal{F} \cup \{E_0\}$. Since $D(H') \subset E_0$ and $H$ is simple, it follows that $D(H')$ is stable.

As $|T'| < |H'|$, a contradiction with the condition of Theorem 3 implies that $\mathrm{Tr}\, H$ cannot be uniform.

**Remark 1** The assertion of Theorem 3 enables also to prove the following result due to Ore [3]:

Let $H$ be a hypergraph without edges of cardinality 1 such that for $T \in \mathrm{Tr}\, H$, $x \in T$, there is a unique edge $E$ for which $E \cap T = \{x\}$. Then $\mathrm{Tr}\, H$ is uniform.

## References


[1] P. Duchet. Sur les hypergraphes invariants. *Discrete Mathematics*, 8(3):269–280, 1974.

[2] P. Gilmore. Cited in C. Berge. *Graphs and hypergraphs.* North-Holland, Amsterdam, 2nd edition, 1973.

[3] O. Ore and T. S. Motzkin. Subsets and subgraphs with maximal properties. *Proceedings of the American Mathematical Society*, 10(6):965–969, 1959.