Discrete convexity : retractions, morphisms and the partition problem

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Résumé — En introduisant de nouveaux morphismes pour les espaces à convexité (abstraits), on montre que pour que l'inégalité de partition générale $p_k \leq (k-1)(p_2 -1)+1$ (célèbre problème de Calder-Eckhoff) soit vraie pour tous les espaces à convexité, il suffit que sa formulation restreinte avec des points distincts soit vraie pour les convexités qui sont à la fois finies, à points convexes et engendrées par une fonction d'intervalle. A l'aide de résultats additionnels (à paraître dans un prochain article), on en déduit que le problème général équivaut à sa formulation restreinte pour les convexités géodésiques des graphes connexes finis (où un ensemble est convexe s'il contient tout plus court chemin entre deux quelconques de ses éléments). Par exemple, pour tout espace à convexité avec nombre de Helly h et $k^{\text{ème}}$ nombre de partition $p_k \geq 4$, on peut construire, un graphe fini G dont la convexité géodésique a pour nombre de Helly h , pour nombre de Radon p_2 et a un $k^{\text{ème}}$ nombre de partition (géodésique) au moins égal à p_k .

Abstract — Introducing certain types of morphisms for general (abstract) convexity spaces, we give several ways for reducing the (Calder-) Eckhoff partition problem to simpler equivalent forms (finite, point-convex, interval spaces; restricted form, i.e. with distinct points). With additional results (to appear in a forthcoming paper) we show how the general problem can be reduced to its restricted version for finite graph-geodetic convexities (in a connected graph a set of vertices is *geodetically convex* if it contains the vertices of any shortest path joining two of its elements). For instance, for every convexity space with Helly number h and partition numbers $p_k \geq 4$, we provide (constructively) a finite connected graph whose (geodetic) convexity has Helly number h, Radon number p_2 and a k-th partition number not less than p_k.

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1 Introduction

A well known theorem by Frucht asserts that every abstract group is the automorphism group of some graph. Informally speaking, graphs constitute a universal model for group structure. Pursuing a systematic study of convex sets in graphs (see [6, 8]), I was led to suspect that graphs also constitute a universal *combinatorial* model for *abstract convexity structures*. In the present paper, I explain how a precise meaning can be given to this guess when Helly and partitions numbers are concerned.

Let us recall that the systematic study of the combinatorial properties of convex sets in R^d, originated in the beginning of the XX^{th} century, has led to a very (too ?) general theory of *abstract convexity spaces*, a theory which reveals to be of independent interest : see [7, 14] for motivation, history and combinatorial aspects , [17] for a precise axiomatic treatment and [19] for a comprehensive recent account. Abstract convexity can be seen as a systematic approach, with geometrical flavour, of algebraic closure system (see [4] for a general theory).

More formally, a *convexity space* is a pair $(X,\mathcal{C})$ where X, the set of *points*, and $\mathcal{C}$, the collection of convex X-subsets, share axioms **(C1)** **(C2)** and **(C3)** here below.

(C1) $[, X \in \mathcal{C}$
(C2) Every intersection of $\mathcal{C}$-members belongs to $\mathcal{C}$.

The intersection of all $\mathcal{C}$-members containing a given X-subset A is named the *convex hull* of A and is usually denoted by $\mathcal{C}(A)$ or by $\langle A\rangle_{\mathcal{C}}$.

(C3) For $x\in X$ and $A \subseteq X$, the property $x \in \mathcal{C}(A)$ implies that $x \in \mathcal{C}(F)$ for some finite subset F of A.

Among various "dimensional" parameters, inspired by the *combinatorial* properties of (ordinary) convex sets in Euclidean spaces, a considerable attention has been given to the "partition numbers".

Definition 1.1 (*partition and Radon numbers*). Let k be an integer, $k\geq 2$. A finite subset A of a convexity space $(X,\mathcal{C})$ is said to be *k-partitionable* if it can be split into k disjoints parts whose convex hulls share a common point. This definition extends easily to multisets. The (unrestricted) *k-th partition-number*, denoted by $p_k(\mathcal{C})$, is defined as the smallest integer p_k such that every multiset with total multiplicity at least p_k is k-partitionable. If no such integer exist, we put $p_k=\infty$. For k=2, p_2 is called the *Radon number* of $\mathcal{C}$ and is frequently denoted by $r(\mathcal{C})$.
For general k, the concept was introduced with Tverberg [18] who proved that, for the ordinary Euclidean convexity in R^d, we have $p_k=(k-1)(d+1)+1$. In particular we have

(1.1) $$p_k\leq (k-1)(p_2-1)+1 ,$$

This inequality, to which we will refer to as the *partition inequality*, holds in various convexity spaces of the (quite vast) literature. Actually, *it holds in all cases where the partition numbers are known*. The question whether the partition inequality holds in *arbitrary convexity spaces*, remains a major unsolved problem of abstract convexity, now known as "*the (Calder-) Eckhoff partition-problem*" [12] or as the *partition conjecture* [19] (the name "*Eckhoff partition-conjecture"* reveals to be somehow

inaccurate : firstly, the problem was raised earlier by Calder for "interval convexity spaces" [3] ; secondly, Eckhoff does not support (1.1) *as a conjecture* in its whole generality).
Jamison (1976, see [15]) proved that p_k is finite for every $k>2$, as soon p_k is finite and gave several conditions [15] under which a convexity space satisfies the partition-inequality. Several attempts to obtain a purely combinatorial proof of Tverberg's theorem have led to simpler proofs but still using intensively geometrical properties of affine real spaces. A recent paper by Roudneff [16] establishes for $\mathbb{R}^4$ a refinement of the partition inequality, conjectured for $\mathbb{R}^n$ by Reay.
The results we present here intend to give more substance to the partition inequality : with the help of an ad hoc definition of retractions and morphisms we obtain several reductions of the problem : by the way, we overcome some very specific difficulties of geometrical type which appears in known proofs of Tverberg's theorem. As an application, we offer a purely metrical approach to the problem, showing how it can be translated in terms of finite graph geodetic convexities.

2 General terminology, Helly and partition dimensions.

An alternative notation for a convexity space $(X,\mathcal{C})$ is $\mathcal{C}[X]$, abridged into $\mathcal{C}$ when no confusion can arise about the ground set. To denote the convex hull of a set A, we prefer the technical fluency of the notation $\langle A\rangle_{\mathcal{C}}$, or simply $\langle A\rangle$ if no ambiguity can occur. For finite sets $\{a\},\{a,b\},\ldots$ the notation is abridged to $\langle a\rangle$, $\langle a,b\rangle$, … If $\langle x\rangle=\{x\}$ for every $x\in X$, the space $(X,\mathcal{C})$ is said to be *point-convex* (or "S1", by reference to one of the weakest "separation axioms" usually considered for convexity spaces). When $\mathcal{C}$ and $\mathcal{D}$ are two convexities on the same ground set X, we say that $\mathcal{C}$ is *coarser* than $\mathcal{D}$, or equivalently that $\mathcal{D}$ is *finer* than $\mathcal{C}$, if $\mathcal{C}\supseteq\mathcal{D}$.

Definition 2.1 (*Y-generated sets*). Let Y be some subset of a convexity space $(X,\mathcal{C})$. A convex set $C\in\mathcal{C}$ is said to be *Y-generated* if C is the $\mathcal{C}$-convex hull of some Y-subset.

Lemma 2.2. A necessary and sufficient criterion for C to be *Y-generated* is :

(2.2) $$C=\langle C\cap Y\rangle_{\mathcal{C}} \qquad \square$$

Note that the intersection of a family of Y-generated convex sets may not be Y-generated.

Topological or algebraic dimensions in Euclidean spaces have no exact analogue in general convexity spaces ; as substitutes, various combinatorial invariants have been considered.

Definition 2.3 (*independence*). Suppose a property Ψ, concerning sets, is defined by some logical formula involving some unary closure operator c which acts on sets. Suppose, in addition, that Ψ is hereditary: $\Psi(B)$ holds as soon as $\Psi(A)$ and $B\subseteq A$. Then, we say that Ψ is an *independence property* (relative to c). A set S is said to be Ψ-*independent.* if $\Psi(S)$ holds. Interpreting c as the convex hull operator in a convexity space $(X,\mathcal{C})$, any independence property determines a *dimensional parameter* $\psi_{\mathcal{C}}$, : $2^X\longrightarrow\mathbb{N}$ (abridged in ψ if no confusion can arise) : the *dimension* $\psi_{\mathcal{C}}$, $(S)$of a X-subset S is, by definition, the maximum size of a Ψ-independent S-subset, with $\psi_{\mathcal{C}},(S)=\infty$ if no such maximum exists. We use also the simpler notation $\psi(\mathcal{C})$ for the maximum value $\psi_{\mathcal{C}},(X)$ of parameter ψ in convexity space $(X,\mathcal{C})$. When no confusion can arise, "$\psi(\mathcal{C})$" is even abridged to "ψ", i.e. we use the same single letter to denote the function and its maximum value.

The following array summarises the definitions of the most usual dimensional parameters involved in the study of the partition problem ; we call them *dimensional parameters of first kind*.

Independence concept	***Defining property (expressed relatively to some X-subset A)***	***Corresponding dimensional parameter and notation***
Convexly independent	$(\forall a \in A)\ a \notin \langle A \setminus a \rangle$	*(convex) dimension* d *or* dim
Helly independent	$\bigcap_{a \in A} \langle A \setminus a \rangle = [$	*Helly dimension* h or h_1
k-Helly independent	$\bigcap_{B \subseteq A,\ \lvert B \rvert = k} \langle A \setminus B \rangle = [$	*restricted k-th Helly dimension* $\dot{h}_k$
Radon independent or *2-partition independent*	If A_1 and A_2 are disjoints A-subsets then $\langle A_1 \rangle \cap \langle A_2 \rangle = [$	*Radon dimension* ρ or ρ_2
k-partition independent	If $A_1, A_2, \dots, A_k$ are disjoints A -subsets then $\bigcap_j \langle A_j \rangle = [$	*restricted k-partition dimension* $\dot{\rho}_k$

Table 2.4 : dimensional parameters of first kind.

Definition 2.5 (*k-partition points*). A k-partition dependent set A is also said to be *k-partitionable*. It admits, by definition, a partition $A_1, A_2, \dots, A_k$ so that $\bigcap_{1 \le j \le k} \langle A_j \rangle \neq [$. Such a partition is called a *(Tverberg-) k-partition* (or, for k=2, a *Radon partition*). Every point in $\bigcap_i \langle A_i \rangle$ is called a *(Tverberg-) k-partition point for A* (or, for k=2, a *Radon point for S*).
The set $\heartsuit_k(A) = \bigcap_{B \subseteq A\ \lvert B \rvert = k} \langle A \setminus B \rangle$ it is named the *k-core of A* : a k-Helly dependent set A has a non empty k-core ; every element of $\heartsuit_k(A)$ is named a *k-core point for A*.

Definition 2.6 (*unrestricted dimensions for multisets*). A *multiset S* of *size m* may be considered as a family of points $S = (s_i)_{i \in I}$ (i.e. as an "*indexed set*"), with $m = \lvert I \rvert$, or as a "*weighted set*" $S = s_1^{m_1}\ s_2^{m_2} \dots s_p^{m_p}$ where each element x_i occurs with multiplicity m_i, with $m = m_1 + m_2 + \dots + m_p$. The definition of the k-Helly (resp. k-partition) independence may be extended in a straightforward manner to multisets. The corresponding dimension parameter is named the (*unrestricted*) *k-Helly* (resp. *k-partition*) *dimension* and is denoted by h_k (resp. ρ_k). Obviously $\dot{h}_k \le h_k$ and $\dot{\rho}_k \le \rho_k$. While those inequalities may be strict in general, note that in every convexity space we have : $\dot{h}_1 = h_1$ and $\dot{\rho}_2 = \rho_2$.

In the literature, the terminology may slightly differs from ours. The *k-th partition number*, also called the *k-th Tverberg number* (or *Radon number* for k=2) exceeds our *k-th partition dimension* by one unit, both in the restricted and unrestricted versions (with the exception of [19], where the *k-th partition number* p_k is our ρ_{k+1}). Our *k-th* Helly dimension h_k is often called the *Helly number of order k*. The same qualification, "*of order k*", is also used for partition parameters.

Lemma 2.7 (Jamison [15]). For $k \geq 2$, the following inequalities hold in any convexity space :

(2.7.1) $h_k = kh_1$

(2.7.2) $\dot{h}_{k-1} \leq \dot{\rho}_k$; $h_{k-1} \leq \rho_k$ (Levi's inequality for k=2)

(2.7.3) $\rho_{kk'} + 1 \leq (\rho_k+1)(\rho_{k'}+1)$

(2.7.4) $\rho_{kk'+1} \leq (\rho_k+1)(\rho_{k'+1})$

3 Subspaces and retracts

A convexity space $(X,\mathcal{C})$ induces in a natural way a convexity space on any X-subset Y by setting :

(3.1) $\mathcal{C}[Y]=\{C\cap Y \,;\, C\in \mathcal{C})$

The space $(Y,\mathcal{C}[Y])$ is named the *convexity subspace* of $(X,\mathcal{C})$ *induced by Y*. Equivalently, we say that $(X,\mathcal{C})$ is *an extension* of $(Y,\mathcal{D})$. Convex hulls in a subspace are given by the following, straightforward but important, identity :

(3.2) $\langle A\rangle_{\mathcal{C}[Y]} = \langle A\rangle_{\mathcal{C}}\cap Y$ for every subset $A\subseteq Y$.

From **(3.2)**, it follows that, passing to subspaces, the convex dimension cannot increase (this is also true for the clique number and the Carathéodory numbers, not studied here).
But for Helly and partition dimensions, the parameters we are interested in here, no general inequality relates the dimension of a subspace to the dimension of the whole space. In order to obtain such inequalities, we will introduce in the next sections special kinds of "morphisms" for convexity spaces. Here below we mention two particular cases in which dimensions cannot increase when passing to subspaces : *convex subspaces* (lemma 3.3) and *retracts* (lemma 3.5).

Lemma 3.3. Let C be a convex set in a convexity space $(X,\mathcal{C})$. Then, for every dimensional parameter of first kind ψ, the function $\psi_{\mathcal{C}[C]}$, relative to the subspace $\mathcal{C}[C]$ coincide with the restriction of function $\psi_{\mathcal{C}}$ to C-subsets. In particular, we have $\psi(\mathcal{C}[C\,])\leq\psi(\mathcal{C})$: the ψ–dimension of a convex subspace is not greater than the dimension of the whole space. □

Definition 3.4 (*retracts*). Let $(X,\mathcal{C})$ be an extension of a convexity space $(Y,\mathcal{D})$ (so, we have $\mathcal{D}^\circ = \mathcal{C}[Y]$). Suppose α is an involutive mapping from X *onto* Y that maps each convex set into itself (resp. each Y-generated convex set into itself). Then, we say that α is a *strong retraction* (resp. a *retraction*) from $(X,\mathcal{C})$ onto $(Y,\mathcal{D})$, and that $(Y,\mathcal{D})$ is a *strong retract* (respectively a *retract*) of $(X,\mathcal{C})$.

Formally, a mapping α from X *onto* Y is a strong retraction (resp. a retraction) iff both the following properties (3.4.1) and (3.4.2) (resp. (3.4.1) and (3.4.3)) are fulfilled :

(3.4.1) $\alpha(y) = y$ for every $y \in Y$

(3.4.2) $\alpha(C) \subseteq C$ for every convex set $C \in \mathcal{C}$.

(3.4.3) $\alpha(C) \subseteq C$ for every Y-generated convex set $C \in \mathcal{C}$.

Condition (3.4.1) may be dropped when $(X,\mathcal{C})$ is point-convex. Note also the following formula :

(3.4.4) $\alpha(C) = C\cap Y$,

which holds if α is a retraction (resp. a strong retraction) and if C is a Y-generated convex set (resp. a convex set).

Lemma 3.5. If $\alpha : X \longrightarrow Y$ is a retraction from $(X,\mathcal{C})$ onto $(Y,\mathcal{D})$, then for every dimensional parameter of first kind, say ψ, we have:

(3.5.1) $\psi(\mathcal{D}) \leq \psi(\mathcal{C})$.

Moreover, if α is a strong retraction, we have

(3.5.2) $\rho_k(\mathcal{C}) = \rho_k(\mathcal{C})$.

(3.5.3) $h_k(\mathcal{D}) = h_k(\mathcal{C})$.

Proof. Let S be a ψ-independent subset in $(Y,\mathcal{D})$. We claim that S is actually ψ-independent in $(X,\mathcal{C})$. We prove the claim for (restricted) k-partition independence ; the reader will easily provide similar proofs for multisets and for other independence properties. If S, considered as a subset in space $(X,\mathcal{C})$, were to admit a k-Tverberg-partition $S_1,S_2,\ldots,S_k$ with partition point σ, then, by (3.2) and (3.4.3), the $\mathcal{D}$-convex hull of each S_j must contain $\alpha(\sigma)$ which therefore constitute a k-partition point for S in $(Y,\mathcal{D})$.
Now let us assume that α is a strong retraction. We prove that every multiset $S = (s_{\mathrm{i}})_{i\in I}$ in X, of size $\rho_k(\mathcal{D})+1$ admits a k-partition for $\mathcal{C}$. Indeed, the multiset $S' = (\alpha(s_{\mathrm{i}}))_{i\in I}$ admits in $(Y,\mathcal{D})$ a k-Tverberg-partition $S'_j = (\alpha(s_i))_{i\in I_j}$, $j = 1,\ldots,k$, with corresponding partition-point σ'. Set $S_j = (s_i)_{i\in I_j}$ for $j = 1,\ldots,k$. By (3.4.2), each multiset $S'_j=\alpha(S_j)$ is included in the convex hull $\langle S_j\rangle_{\mathcal{C}}$, hence in $\langle S_j\rangle_{\mathcal{D}} = \langle S_j\rangle_{\mathcal{C}}\cap Y$. contains $S'_j=\alpha(S_j)$. Since the set $\langle S_j\rangle_{\mathcal{D}}$ is a $\mathcal{D}$-convex that contains $S'_j=\alpha(S_j)$, it contains $\langle S'_j\rangle_{\mathcal{D}}$, hence σ', which therefore constitutes a k-partition point for S. Assertion (3.5.2) follows.
The proof of (3.5.3) would be similar. □

Remark 3.6. Inequality (3.5.1) may be strict for restricted Helly or partition dimensions, even when strong retracts are considered. It also may happen than, under a strong retraction, the image of some set which is ψ-independent in the original space fails to be ψ-independent in the retract space.

4 Morphisms

Various morphisms are usually considered for abstract convexity spaces. To reach our goal concerning the partition problem, we are led to weaken some classical notions (compare with "convex-preserving functions" or with "convex to convex functions" [19]).

Definition 4.1 (*Morphisms*). Let $(X,\mathcal{C})$ and $(Y,\mathcal{D})$ be two convexity spaces. A mapping $\mathrm{f} : X \longrightarrow Y$ is called a *morphism* (respectively, a *weak morphism*) from $(X,\mathcal{C})$ onto $(Y,\mathcal{D})$ if the following property (4.1.1) (resp. (4.1.2)) holds :

(4.1.1) $x \in \langle \mathrm{f}^{-1}(T)\rangle_{\mathcal{C}}$ implies $\mathrm{f}(x) \in \langle T\rangle_{\mathcal{D}}$, for every Y-subset T .

(4.1.2) $x \in \langle \mathrm{f}^{-1}(T)\rangle_{\mathcal{C}}$ implies $\mathrm{f}(x) \in \langle T\rangle_{\mathcal{D}}$, for every Y-subset T with at least two points.

When f is such a *morphism* (resp., a *weak morphism*), the space $(Y,\mathcal{D})$ is said to be *homologous* (resp., *weakly homologous*) to $(X,\mathcal{C})$ via f.

Significant examples of morphisms and weak morphisms will be given in subsequent sections. Those concepts are motivated by the following important lemma.

Lemma 4.2 — Let be given two convexity spaces $(X,\mathcal{C})$, $(Y,\mathcal{D})$ and a mapping $f : X \longrightarrow Y$. Suppose that at least one of the following conditions is fulfilled : (a) f is a morphism, (b) f is a weak morphism and (X,C) is point-convex. Then the following inequality holds for every first kind parameter ψ:

(4.2) $\psi(\mathcal{D}) \leq \psi(\mathcal{C})$

Proof. We give the proof for the unrestricted k-partition dimension ; proofs of other inequalities would be similar (and easier). Let us consider in $(Y,\mathcal{D})$ a multiset $S' = (s'_i)_{i\in I}$ of size ρ_k+1. We prove that S' is k-partitionable. For every index i, we choose a preimage s_i of s'_i (with other words, we have $f(s_i) = s'_i$), The s_i 's are chosen so that $s_\lambda=s_\mu$ as soon as $s'_\lambda = s'_\mu$, i.e. the multiplicity of each s_i is the same in multiset $S = (s_i)_{i\in I}$ than the multiplicity of s'_i in S'. Multiset S has size ρ_k+1, hence admits in $(X,\mathcal{C})$ a k-Tverberg-partition $S_1,S_2,\ldots,S_k$ with partition point σ. Let denote by $S'_1,S'_2,\ldots,S'_k$ the corresponding partition of multiset S'.

The remaining of the proof is split into two cases, according to hypotheses (a) or (b) of our proposition ; in each case we apply Definition 4.1, observing that $\langle S_j\rangle_\mathcal{C} \subseteq \langle f^{-1}(S'_j)\rangle_\mathcal{C}$.

• Case (a) : f is a morphism. Applying (4.1.1), we obtain $f(\sigma)\in\langle S'_j\rangle_\mathcal{D}$ for every j.

• Case (b) : f is a weak morphism and $(X,\mathcal{C})$ is point-convex. We see, applying (4.1.2), that $f(\sigma)$ belongs to $\langle S'_j\rangle_\mathcal{D}$ for every index j such that the *set* supporting the multiset S'_j has at least two elements. For any other index j , multiset S'_j is supported by a single element, hence, by the definition of the s_i 's, so is S_j . Thus, by the point-convexity of $(X,\mathcal{C})$, we have $S_j = \langle S_j\rangle_\mathcal{C} =\{\sigma\}$. Therefore $f(\sigma)\in\langle S'_j\rangle_\mathcal{D}$ holds for every j.

In both cases $f(\sigma)$ constitutes a k-partition point for S', as required. □

Remark 4.3. For weak morphisms, only weaker inequalities can be proved in general ; for instance, we have $\rho_k(\mathcal{D}) \leq \rho_k(\mathcal{C})+k-1$.

5 Image and quotient of convexity spaces

Let $(X,\mathcal{C})$ be a convexity space and suppose we have a mapping f from X *onto* a set Y.

Definition 5.1 *(Image space)* **.** Endowing Y with the coarsest convexity that contains, as convex sets, all Y-subsets of the form $f(C)$ for $C\in\mathcal{C}^\circ$, we obtain a convexity space we call the *image* of $(X,\mathcal{C})$ *under* f ; we denote it by $(Y,f\mathcal{C})$.

As easily seen, the $f\mathcal{C}$-convex hull of an arbitrary Y-subset T is given by :

(5.1.1) $\langle T\rangle_{f\mathcal{C}} = \bigcap \{f(C) \ ; \ C\in\mathcal{C} \text{ and } T \subseteq f(C) \}$ □

We point out also the following inclusion which readily follows from Definition 5.1 :

(5.1.2) For every $S \subseteq X$, we have $\langle f(S)\rangle_{f\mathcal{C}} \subseteq f(\langle S\rangle_\mathcal{C})$ □

Remark 5.2. If $f : X\longrightarrow Y$ is a retraction of $(X,\mathcal{C})$ onto $(Y,\mathcal{C}[Y])$, then $f\mathcal{C}^\circ= \mathcal{C}[Y]$. (the easy proof is left to the reader).

In general, equality does not hold in (5.1.2) (even if f is a retraction). For a mapping f to be a morphism from $(X,\mathcal{C})$ to $(Y,\mathrm{f}\mathcal{C})$ additional assumptions are needed.

Definition 5.3 (*faithful and conform mappings*). Let be given a convexity space $(X,\mathcal{C})$ and f : . a mapping from X *onto* another set Y. Mapping f and the corresponding image space $(Y,\mathrm{f}\mathcal{C})$ are said to be $\mathcal{C}$-*faithful* (resp. *weakly* $\mathcal{C}$-*faithful,* $\mathcal{C}$-*conform, weakly* $\mathcal{C}$-*conform*), when the following condition (5.2.1) (resp. (5.2.2), (5.2.3), (5.2.4)) is fulfilled, for every $x,x' \in X$, $C \in \mathcal{C}$, $S \subseteq X$:

(5.3.1) $\langle \mathrm{f}(S)\rangle_{\mathrm{f}\mathcal{C}} = \mathrm{f}(\langle S\rangle_{\mathcal{C}})$.

(5.3.2) $\langle \mathrm{f}(S)\rangle_{\mathrm{f}\mathcal{C}} = \mathrm{f}(\langle S\rangle_{\mathcal{C}})$, as soon as $|\mathrm{f}(S)| \geq 2$.

(5.3.3) If $\mathrm{f}(x) = \mathrm{f}(x')$, then $x \in C$ implies $x' \in C$.

(5.3.4) If $\mathrm{f}(x) = \mathrm{f}(x')$ and if $|C| \geq 2$, then $x \in C$ implies $x' \in C$.

The next lemmas will allow us to apply inequalities of lemma 4.2 to faithful or conform images.

Lemma 5.4. Let $(X,\mathcal{C})$ be an arbitrary convexity space. Every $\mathcal{C}$-faithful (resp. weakly $\mathcal{C}$-faithful) mapping f from X onto Y is a morphism (resp. a weak morphism) from $(X,\mathcal{C})$ onto the image space $(Y,\mathrm{f}\mathcal{C})$.

Proof. If f fulfils (5.3.1), then for $T \subseteq Y$ we have : $\langle T\rangle_{\mathrm{f}\mathcal{C}} = \langle \mathrm{f}(\mathrm{f}^{-1}(T))\rangle_{\mathrm{f}\mathcal{C}} = \mathrm{f}(\langle \mathrm{f}^{-1}(T)\rangle_{\mathcal{C}})$; property (4.1.1) follows. Similarly (5.3.2) implies (4.1.2). □

Lemma 5.5. A conform (resp. weakly conform) mapping is faithful (resp. weakly faithful).

Proof. Given a weak $\mathcal{C}$-conform mapping $\mathrm{f} : X \longrightarrow Y$, let S be an arbitrary subset of X, which, without loss of generality, we suppose non-empty. We have (see (5.1.2)) : $\langle \mathrm{f}(S)\rangle_{\mathrm{f}\mathcal{C}} \subseteq \mathrm{f}(\langle S\rangle_{\mathcal{C}})$. To examine the converse inclusion, let us consider an arbitrary $\mathcal{C}$-convex set C such that $\mathrm{f}(S\rangle \subseteq \mathrm{f}(C)$. To put it differently, we see that C contains, for every $s \in S$, such an element s^* that $\mathrm{f}(s^*\rangle = \mathrm{f}(s)$. We consider two cases.

- Case 1 : f is conform.
- Case 2 : $|\mathrm{f}(S)| \geq 2$. Thus $|\mathrm{f}(C)| \geq 2$, hence $|C| \geq 2$.

Applying (5.3.3) in the first case, or (5.3.4) in the second, we see that C contains all elements of S, hence, by convexity, it contains $\langle S\rangle_{\mathcal{C}}$. So, in each case, any $\mathcal{C}$-convex set whose image contains $\mathrm{f}(S\rangle$, contains $\langle S\rangle_{\mathcal{C}}$. Using (5.1.1), we obtain respectively (5.3.1) or (5.3.2) in cases 1 or 2. □

Conform and faithful mappings provides more inequalities than those of Lemma 4.2.

Lemma 5.6 Let $(Y,\mathrm{f}\mathcal{C})$ be the image of a convexity space $(X,\mathcal{C})$ by a weakly-conform mapping f from X onto Y. Then :

(5.6.1) $\rho_2(\mathcal{C}) \leq \rho_2(\mathrm{f}\mathcal{C})$, if $\rho_2(\mathrm{f}\mathcal{C}) \geq 2$.

(5.6.2) $h_1(\mathcal{C}) \leq h_1(\mathrm{f}\mathcal{C})$, if $h_1(\mathrm{f}\mathcal{C}) \geq 2$.

If, in addition, f is conform, then :

(5.6.3) $\rho_k(\mathcal{C}) = \rho_k(\mathrm{f}\mathcal{C})$ for all $k \geq 2$.

Proof. Assuming $\rho_2(\mathrm{f}\mathcal{C}) \geq 2$, we prove (5.6.1) under the form $\dot{\rho}_2(\mathcal{C}) = \rho_2(\mathcal{C}) \leq \rho_2(\mathrm{f}\mathcal{C})$. Given $S \subseteq X$, a set of $r = \rho_2(\mathrm{f}\mathcal{C})+1$ elements, we have to prove that S admits a Radon partition in $(X,\mathcal{C})$. Since the multiset $S' = (\mathrm{f}(s))_{s\in S}$ has size r, it admits in $(Y,\mathrm{f}\mathcal{C})$ a Radon partition S'_1, S'_2 with Radon-point σ'. This Radon partition corresponds to a partition S_1, S_2 of the index set S. Since $\langle S_j\rangle_{\mathcal{C}} \subseteq \langle \mathrm{f}^{-1}(S'_j)\rangle_{\mathcal{C}}$, each convex hull $\langle S_j\rangle$ contains some point σ_j of $\mathrm{f}^{-1}(\sigma')$. Since $\rho_2(\mathrm{f}\mathcal{C}) \geq 2$, one of the set S_1 or S_2, say S_2, is not reduced to a single element. Hence, applying (5.3.4), $\langle S_2\rangle$ contains σ_1 which therefore constitutes a Radon point for S, as required.

The proof of (5.6.2) is similar and is left to the reader.

Now we establish (5.6.3), assuming f is conform. Applying lemmas 5.5 and 5.4 and proposition 4.2, we obtain $\rho_k(\mathrm{f}\mathcal{C}) \leq \rho_k(\mathcal{C})$. Thus, it remains to prove that any multiset $S = (s_\mathrm{i})_{i\in I}$ constituted with $\rho_k(\mathrm{f}\mathcal{C})+1$ elements of X admits a k-Tverberg-partition in $(X,\mathcal{C})$. Observe that, in $(Y,\mathrm{f}\mathcal{C})$, the multiset $S' = (\mathrm{f}(s_\mathrm{i}))_{i\in I}$ admits a k-Tverberg-partition $S'_j = (\mathrm{f}(s_i))_{i\in I_j}$, $j = 1,\ldots,k$, with corresponding partition-point σ'. Setting $S_j = (s_i)_{i\in I_j}$, for every j, we note that $\langle S_j\rangle_{\mathcal{C}} \subseteq \langle \mathrm{f}^{-1}(S'_j)\rangle_{\mathcal{C}}$: hence, each convex hull $\langle S_j \rangle_{\mathcal{C}}$ contains some point σ_j in $\mathrm{f}^{-1}(\sigma')$. Since f is conform, each $\langle S_j\rangle$ contains all the σ_j's. Every point of the convex hull $\langle\{\sigma_j\,;\,1\leq j\leq k\}\rangle_{\mathcal{C}}$ is therefore a k-partition point for S, as requested.

□

Remark 5.7. Proposition 5.6 is essentially the best possible. Rather crude examples shows that for $k\geq 2$ the k-th Helly dimension and the k-th restricted partition dimension of a *conform* image space may be strictly smaller than in the original space ; also, we may have $\rho_k(\mathrm{f}\mathcal{C}) < \rho_k(\mathcal{C}) \leq k$ for a weakly-faithful image of a point-convex space. Notice that a conform mapping which is one to one (this is the case when $(X,\mathcal{C})$ is point-convex) is actually an isomorphism.

Definition 5.8 (*quotient spaces*). When R is an equivalence relation on X, the image of the convexity space $(X,\mathcal{C})$ by the quotient map $\mathrm{f}_\mathrm{R} : X \longrightarrow X/\mathrm{R}$ is named the *quotient convexity space* of $(X,\mathcal{C})$ by R and is denoted by $(X/\mathrm{R},\mathcal{C}/\mathrm{R})$. We will use for quotient spaces the same qualifying terms we use for their associated quotient maps.

6 Reduction to finite spaces

Passing to finite subspaces may alter drastically the value of combinatorial parameters of an infinite convexity space. For instance, there exist infinite denumerable spaces with (first) Helly number 2 in which every finite subspace with at least n points has Helly dimension n. In order to simulate by finite spaces the dimension parameters of general convexity spaces, we are going to construct a suitable quotient space.

Theorem 6.1 Let Π be a finite list of dimensional parameters of first kind (see table 2.4). Every convexity space admits a "*finite model for* Π". More precisely, if $(X,\mathcal{C})$ is a convexity space whose ψ-dimension is finite for every $\psi\in\Pi$, then there exist a finite convexity space $(X_\Pi,\mathcal{C}_\Pi)$ so that $\psi(\mathcal{C}) = \psi(\mathcal{C}_\Pi)$ for every $\psi\in\Pi$.

Proof — The construction proceeds in three steps.

We start with a finite set F of X, we choose sufficiently large to contain, for every parameter ψ in Π, some ψ-independent subset of maximum size. From formula (3.2) and Lemma 3.3, it follows that, for every ψ in Π, the space $\mathcal{C}[X]$ and its convex subspace $\mathcal{C}[\langle F\rangle_{\mathcal{C}}]$ have the same ψ-dimension.

We endow the set $\langle F\rangle_{\mathcal{C}}$ with a coarser convexity $\mathcal{F}$, defined as follows : a subset is $\mathcal{F}$- convex if it is the intersection of F-generated $\mathcal{C}$-convex sets. Obviously, the convex hulls of F-subsets coincide in $\mathcal{C}$ and in $\mathcal{F}$, and the identity map from the convex subspace $(\langle F\rangle_{\mathcal{C}}, \mathcal{C}[\langle F\rangle_{\mathcal{C}}])$ onto $(\langle F\rangle_{\mathcal{C}}, \mathcal{F})$ is a morphism. Hence Proposition 4.2 applies and we have $\psi(\mathcal{F}) \leq \psi(\mathcal{C}[\langle F\rangle_{\mathcal{C}}]) = \psi(\mathcal{C})$.

Now, we construct a $\mathcal{F}$-conform finite quotient space. Let $F_1, F_2,\ldots, F_N$ be the list of all non-empty F-subsets. For $1\leq p\leq N$, and $x \in \langle F\rangle_{\mathcal{C}}$ we put $\phi_p(x) = 1$ if $x\in\langle F_p\rangle$ and $\phi_p(x) = 0$ otherwise. The sequence $\phi_1(x), \phi_2(x),\ldots, \phi_N(x)$, consisting of 0's and 1's, is referred to as the *profile* of x. Let us introduce an equivalence relation R on $\langle F\rangle_{\mathcal{C}}$ as follows : two points of $\langle F\rangle_{\mathcal{C}}$ are R-*equivalent*. either if they are equal, or if both of them lie outside of F and have the same profile. As easily checked, the corresponding quotient function f, mapping each point to its R-equivalence class, is a $\mathcal{F}$-conform morphism. Hence, by proposition (4.2), we have $\psi(\mathcal{F}/\mathrm{R}) \leq \psi(\mathcal{F})$.

As readily seen, the convex hulls of F-subsets coincide in $\mathcal{C}$ and in $\mathcal{F}/\mathrm{R}$, hence, visualising the defining property of ψ-independence (see Table 2.4), we conclude : any F-subset which is ψ-independent in $\mathcal{C}$ is also, when considered as a subset of $\langle F\rangle_{\mathcal{C}}/\mathrm{R}$, ψ-independent in $\mathcal{F}/\mathrm{R}$. Hence $\psi(\mathcal{C}) \leq \psi(\mathcal{F}/\mathrm{R})$. Combining with the preceding inequalities we obtain $\psi(\mathcal{F}) = \psi(\mathcal{C})$. The quotient space $(\langle F\rangle_{\mathcal{C}}/\mathrm{R}, \mathcal{F}/\mathrm{R})$ is a finite model of $\mathcal{C}$ for Π, as required . □

7 Reduction to finite point-convex spaces

Let assume that $(X,\mathcal{C})$ is a *finite* convexity space. Say that a point y is *regular* if its convex hull $\langle y\rangle$ is inclusion-minimal in $\mathcal{C}$.. Since X is finite, every non empty convex set contains some regular point. To every point x in X, we associate a regular point $\mathrm{f}(x)$ in $\langle x\rangle_{\mathcal{C}}$ as follows :

1) For each $x \in X$, we choose a minimal convex set $\mu(x)$ in $\langle x\rangle$.

2) For each minimal convex set M , we choose a (regular) point $\mathrm{p}(M)$ in M .

Now, put $\mathrm{f} = \mathrm{p} \circ \mu$. Any mapping f constructed in such a way, is named a *simplification* of $(X,\mathcal{C})$.

As easily seen, f is a *strong retraction* from $(X,\mathcal{C})$ onto the subspace $(\mathrm{f}(X),\mathcal{C}[\mathrm{f}(X)])$ which coincide (cf. Remark 3.2) with the image space $(\mathrm{f}(X), \mathrm{f}\mathcal{C})$; any strong retract of that form is called a *simplified form* of X.

Proposition 7.1. Any convexity space obtained from a finite convexity space $(X,\mathcal{C})$ by simplification is point-convex and has, for all $k.\geq 2$, the same k-partition dimension as $(X,\mathcal{C})$. □

Proof. Every regular point of $(X,\mathcal{C})$ is contained in a unique minimal convex set. The point-convexity of any simplified form follows. Our proposition then follows from lemma 3.5. □

8 Reduction to Interval convexities

A very general way to define a convexity space is to consider a family $\Theta=(\theta_i)$ of operators, each θ_i being of finite arity t_i :

$$\theta_i : X^{t_i} \longrightarrow X.$$

In the convexity space *generated by* Θ, convex sets are defined as those X-subsets which are closed under every operator θ_i. Convexities generated by a single 2-ary operator play a prominent role in Abstract Convexity. They are called *interval convexities* (Calder [3]). Any 2-ary operator generating such a convexity $\mathcal{C}$ is named an *interval function for* $\mathcal{C}$. Let us recall some basic known results on interval functions and interval convexity spaces (see [17] for more details).

Let $\mathcal{C}$ denote the interval convexity generated by an interval function $\mathrm{I} : X{\times}X \longrightarrow X$. The $\mathcal{C}$-convex hull $\langle A\rangle$ of an arbitrary subset A of X may be expressed in terms of iterated functions I^k , defined recursively as follows:

(8.1.1) $\mathrm{I}^0(A) = A$

(8.1.2) $\mathrm{I}^{k+1}(A) = \mathrm{I}(\mathrm{I}^k(A)\times \mathrm{I}^k(A))$

Setting $\mathrm{I}^{\omega}(A) = \bigcup_{k\in\mathbb{N}} \mathrm{I}^k(A)$, we have :

(8.2) $\langle A\rangle = \mathrm{I}^{\omega}(A)$ for any $A \subseteq X$.

Remark 8.3. As pointed out by Calder [3], any interval convexity $\mathcal{C}$ may be generated by the particular interval function (x,y) $\langle x,y\rangle_{\mathcal{C}}$.

The following result suggests that, in some sense, Abstract Convexity may be seen as a theory of interval functions.

Lemma 8.4 (Burris [2], see [17]). Every convexity space of cardinal κ (respectively, of finite cardinal) is a subspace of some interval convexity space $\overline{S}$ of cardinal κ (respectively, of finite cardinal).

Burris's construction possesses interesting properties which have remained, up to now, unnoticed : a large part of the structure of the original convexity, and, namely, some combinatorial parameters, are preserved when extending the space to an interval space. Below, we are going to refine Burris's idea in order to be able to manage the investigations on first kind combinatorial parameters of general (point-convex) convexity spaces in the restricted setting of (point-convex) interval convexities.

Proposition 8.5. Every point-convex convexity space S of infinite cardinal κ (respectively, of finite cardinal) admits an canonical extension $\mathrm{S}^{\sim}$ of cardinal κ (respectively, of finite cardinal) with the following properties

(i) $\mathrm{S}^{\sim}$ is a point-convex interval convexity space

(ii) S is a retract of $\mathrm{S}^{\sim}$.

(iii) For every dimensional parameter of first kind ψ, we have $\psi(\mathrm{S}^{\sim})= \psi(\mathrm{S})$.

Proof *(sketched).* Let's be given $\mathrm{S}=(X,\mathcal{C})$ an arbitrary convexity space. Let $X^{\bullet}$ denote the set of finite non-empty X-subsets. For every $E\in X^{\bullet}$, let x_E be a new symbol. For every $Y\subseteq X$, we set :

(8.5.1) $Y^{\sim} = \{x_E \mid E\subseteq Y, E \text{ is finite and non empty}\}$

In particular, identifying x and $x_{\{x\}}$, X is canonicaly embedded in $X^{\sim}$. We define an interval function $\mathrm{I} : X^{\sim}{\times}X^{\sim} \longrightarrow X^{\sim}$ by setting:

(8.5.2) $\mathrm{I}(x_E,x_E) =\{x_E\}$ for $E\in X^{\bullet}$

(8.5.3) $\mathrm{I}(x_E,x_F) = \{x_E,x_F,x_{E\cup F}\} \cup \langle E\cup F\rangle_{\mathcal{C}}$ for $E ,F \in X^{\bullet}$, $E \neq F$

The interval function I generates a point-convex interval convexity space $S^{\sim}=(X^{\sim},\mathcal{C}^{\sim})$. Denoting respectively by $\langle . \rangle$ and $\langle . \rangle^{\sim}$ the convex hulls in $\mathcal{C}$ and in $\mathcal{C}^{\sim}$, let us examine the convex hull in $\mathcal{C}^{\sim}$ of an arbitrary $X^{\sim}$-subset S. The *support* $\underline{S}$ of S is the X-subset defined by :

(8.5.4) $\underline{S} = \bigcup \{E \subseteq X \mid x_E \in Z)$.

Note that $\underline{S}$ equals S if $S \subseteq X$. Considering the iterated functions I^k (see (8.1)) and using (8.4.3), we observe that, if S is *non-trivial* (i.e. has cardinality at least 2), then $\mathrm{I}^{k+1}(S)$ contains all sets of the form $\langle T \rangle$ for every subset T with at most k elements of $\underline{S}$. From this observation, follows the formula :

(8.5.5) $\langle S \rangle^{\sim} = (\langle \underline{S} \rangle)^{\sim}$ for every non trivial $X^{\sim}$-subset S.

That formula has the following immediate consequences :

(8.5.6) $\langle A \rangle^{\sim} = (\langle A \rangle)^{\sim}$ for any $A \subseteq X$ (even for single element sets).

(8.5.7) $X \cap \langle S \rangle^{\sim} = \langle \underline{S} \rangle$ for every non trivial $X^{\sim}$-subset S.

In particular, we have $X \cap \langle A \rangle^{\sim} = \langle A \rangle$ for any $A \subseteq X$: with other words, $(X,\mathcal{C})$ is a convexity subspace of $(X^{\sim},\mathcal{C}^{\sim})$. Part (i) of lemma 8.4 is established. □

To prove part (ii), let us choose (see remark 8.5 below) in each finite non-empty X-subset E some element $x(E)$. By (8.5.5), every non-trivial $\mathcal{C}^{\sim}$-convex set is X-generated. Hence, the mapping $\phi : X^{\sim} \longrightarrow X$ defined by $\phi(x_E) = x(E)$ constitutes a retraction from $(X^{\sim},\mathcal{C}^{\sim})$ onto $(X,\mathcal{C})$. □

Therefore, Lemma 3.5 applies : so, to establish (iii), it suffices to prove that for every dimensional parameter of first kind ψ, any ψ-independent subset in $(X,\mathcal{C})$ is also ψ-independent in $(X^{\sim},\mathcal{C}^{\sim})$. We give here below the proof for the restricted k-partition-independence, leaving to the reader the care to adapt it to other parameters.

We proceed by contradiction. Let us suppose S is a finite k-partition-independent set in $(X,\mathcal{C})$ but admits in $(X^{\sim},\mathcal{C}^{\sim})$ a k-partition $S_1,S_2,\ldots,S_k$ with partition point x_T, where T denotes some finite non-empty X-subset. By (8.5.6), we have $T \subseteq \langle S_i \rangle$ for every i, hence any element of T constitutes a partition point for S : a patent contradiction. □ □

Remark 8.6. In the proof above, the use of lemma 3.5 and of Choice Axiom can be avoided : inequalities involving dimensional parameters can be given a direct proof, following the same lines as for lemma 3.5.

9 Convexity simulation via graph geodesics

See [1] for general notions of Graph Theory. A set of vertices of a connected graph is *geodetically convex* if it contains the vertices of any shortest path joining two of its elements. The collection of geodetically convex sets of a given graph $G = (V,E)$ is called the *geodetic convexity of* G. Convexities arising in such a way are referred to as *graph geodetic convexities*.

Obviously, the geodetic convexity of a graph $G=(V,E)$ is a point-convex interval convexity generated be the following *geodetic-interval function* I_G :

$$\mathrm{I}_G(v,w) = \{z \,;\, z \text{ is on some shortest path joining } v \text{ to } w\}$$

As announced in [8], *graph geodetic convexities* constitute a "good model" for dimensional parameters of general abstract convexity spaces. The results here below give a precise substance to that assertion for dimensional parameters of first kind.

Theorem 9.1. [10] Every point-convex interval convexity is a retract of some graph geodetic convexity. Moreover there is a canonical way to construct such an extension ; this canonical extension is finite whenever the original convexity is so.

Sketch of proof : Let $(X,\mathcal{C})$ a point-convex interval convexity space with interval function $\mathrm{I}(x,y) = \langle x,y\rangle_{\mathcal{C}}$ (see Remark 8.3). Let us introduce the following sets

(9.1.1) $X_1 = \{(x,y) \in X\times X\ ;\ x \neq y\}$.

(9.1.2) $X_2 = \{(x,y,z) \in X\times X\times X\ ;\ x,y,z$ are distinct and $z \in \mathrm{I}(x,y)\}$.

Without loss in generality, we suppose that X, X_1 and X_2 are mutually disjoint sets. To avoid technicalities, we adopt the more simple notation xy (resp. xyz) for an element (x,y) of X_1 (resp. for an element (x,y,z) of X_2). We now define a graph $\mathbf{G} = (X^*,E)$ as follows.

(9.1.3) The vertex set of $\mathbf{G}$ is $X^* = X\cup X_1\cup X_2$.

(9.1.4) E, the edge-set of $\mathbf{G}$, is constituted by the following edges :
$[x,xy]$ and $[x,yx]$, for every distinct points x,y of X
$[xy,xyz]$, $[xy,yxz]$, $[z,xyz]$, for every distinct points x,y,z of X such that $z \in \mathrm{I}(x,y)$.

Note that, by (9.1.4), $[yx,yxz]$, $[yx,xyz]$ and $[z,yxz]$ are also edges of $\mathbf{G}$.

In addition, a mapping $\xi : X^* \longrightarrow X$ is defined by :

(9.1.5) $\xi(x) = x$, $\xi(xy) = x$, $\xi(xyz) = z$ for every $x \in X$, $xy \in X_1$, $xyz \in X_2$.

Let us denote by $\mathcal{C}^*$ the geodetic convexity in $\mathbf{G}$, generated by the geodetic interval function I*. We denote by $\langle . \rangle$ and $\langle . \rangle^*$ the convex hulls in $\mathcal{C}$ and in $\mathcal{C}^*$. Intuitively, I* "simulates" the interval function that generates $\mathcal{C}$; formally, we have :

- $\mathrm{I}^*(x,x) = \{x\}$, for $x \in X$
- $\mathrm{I}^*(x,y) = \{x,xy,yx,y\}$, for $x,y \in X$, $x\neq y$.
- $\mathrm{I}^*(xy,yx) = \{xy,yx,x,y\} \cup \bigcup\limits_{z\in \mathrm{I}(x,y)} \{xyz,yxz\}$, for xy, $yx \in X_1$.
- $\mathrm{I}^*(xyz,yxz) = \{xyz,yxz,z,xy,yx\}$ for $xyz,yxz \in X_2$.

Hence $\langle x,y\rangle \subseteq \langle x,y\rangle^*$ for $x,y \in X$. A detailed analysis of the structure of geodetic convex sets in $\mathbf{G}$ (see[10]) leads to the following salient facts :

Lemma 9.2. We have $\langle x,y\rangle^{*\circ} = \langle x,y\rangle \cup \langle x,y\rangle_1 \cup \langle x,y\rangle_2$, where :
$\langle x,y\rangle_1 = \{uv \in X_1\ ;\ u,v \in \langle x,y\rangle\}$, and $\langle x,y\rangle_2 = \{uvw \in X_2\ ;\ u,v \in \langle x,y\rangle$ and $w \in \langle u,v\rangle\}$. □

Lemma 9.3. For every $A\subseteq X$, we have $\langle A\rangle = X \cap \langle A\rangle^*$. □

Lemma 9.4. For every $S\subseteq X^*$, the hypotheses $\boldsymbol{x}\in\langle S\rangle^*$ and $\boldsymbol{x}\in S$ imply $\xi(\boldsymbol{x})\in\langle S\rangle^*$. □

By Lemma 9.3, $(X^*,\mathcal{C}^*)$ is an extension of $(X,\mathcal{C})$. Then, by Lemma 9.4, mapping ξ is a retraction from $(X,\mathcal{C}^*)$ onto $(X,\mathcal{C})$. Theorem 9.1 follows. □ □

Definition 9.5 (*canonical graph-geodetic extensions*)**.** Graph $\mathbf{G}$, as defined above, is named the (*canonical*) *graph associated to* $\mathcal{C}^\circ$; the corresponding convexity space, $(X^*,\mathcal{C}^*)$, is named the (*canonical*) *graph-geodetic extension of* $(X,\mathcal{C})$; mapping ξ is the *canonical retraction* from $(X^*,\mathcal{C}^*)$ onto $(X,\mathcal{C})$.

Theorem 9.6 [10]**.** The dimensional parameters of the graph-geodetic extension (X^*,$\mathcal{C}^*$) of any point-convex interval convexity space (X,$\mathcal{C}$) satisfy the following inequalities.

(9.6.1) $\psi(\mathcal{C}) \leq \psi(\mathcal{C}^*)$ for every parameter ψ of first kind.

(9.6.2) $\rho_2(\mathcal{C}^*) = \rho_2(\mathcal{C})$, as soon as $\rho_2(\mathcal{C})\geq 3$.

(9.6.3) $h_1(\mathcal{C}^*) = h_1(\mathcal{C})$, as soon as $h_1(\mathcal{C})\geq 2$.

***proof** (outline).* Since (X,$\mathcal{C}$) is a retract of (X^*,$\mathcal{C}^*$), inequality (9.6.1) results of Lemma 3.5.
In the spirit of the proof given above for (5.6.1), let S^* denote a multiset in X^* with size $\rho_2(\mathcal{C})+1$. The multiset $S = (\xi(s))_{s\in S^*}$, image of S^* by the canonical retraction ξ, admits a Radon partition S_1, S_2 in (X,$\mathcal{C}$). A careful case by case examination of the sets S_1, S_2 and of their preimages $\xi^{-1}(S_1)$, $\xi^{-1}(S_2)$, together with the use of lemma 9.2, lead to the conclusion that S^* is Radon-dependent in (X^*,$\mathcal{C}^*$), in all cases, except, possibly, if S^* has 3 elements or less.
Equality (9.6.2) follows. Similar arguments provide (9.6.3). □

10 Equivalent forms for the partition problem

For topological reasons, if a multiset of $\mathbb{R}^n$ is not k-partitionable, any multiset (thus, *a fortiori,* any set) obtained by small pertubations of its points is still not k-partitionable. Hence, to prove Tverberg's theorem it suffices to consider points in general positions. We point out that some similar phenomenon occurs when attacking the partition problem in general convexity spaces : the consideration of set of distinct points would suffice to settle the conjectured inequality.

Proposition 10.1. If the partition inequality holds for restricted partition numbers in every point-convex convexity space, then it holds in every convexity space.

Proof. Assume inequality $\dot{\rho}_k \leq (k\text{-}1)\rho_2$ holds in every point-convex convexity space. To establish the proposition, it suffices, by Proposition 7.1, to prove that the partition inequality holds for point-convex spaces. Let (X,$\mathcal{C}$) be a point-convex convexity space with finite Radon dimension ρ. In (X,$\mathcal{C}$), let us consider an arbitrary multiset $S=s_1^{m_1}\ s_2^{m_2}\ \ldots\ s_p^{m_p}$, of size $m = m_1+m_2+\ldots+m_p =$ (k-1)ρ+1. Without loss in generality, we suppose the s_i's are all distinct and occur with positive multiplicities. We have to prove that S is k-partitionable.
To each point s_i , let us associate m_i -1 new and different elements $s_i^{(2)}$, $s_i^{(3)}$,…, $s_i^{(m_i)}$. Adding the new elements to X, we obtain an extended set X^S, where every element $s_i^{(j)}$ may be identified to the j-th occurrence of s_i in the set S (for convenience, we set $s_i^{(1)} = s_i$). On X^S we define a convexity structure $\mathcal{C}^S$, extending $\mathcal{C}$ as follows :

$$\mathcal{C}^S = \{\mathrm{f}^{-1}(C) \ ;\ C\in\mathcal{C}\} \cup \{\{s_i^{(j)}\}\ ;\ 1\leq i\leq p\ ,\ 2\leq j\leq m_i\ \}$$

where the mapping $\mathrm{f} : X^S \longrightarrow X$ is defined by $\mathrm{f}(x) = s_i$ for $x = s_i^{(j)}$ ($1\leq i\leq p$, $1\leq j\leq m_i$) and $\alpha(x) = x$ otherwise. Obviously, f is a retraction from (X^S,$\mathcal{C}^S$) onto (X,$\mathcal{C}$), hence we have f$\mathcal{C}^S$=$\mathcal{C}$ (by Remark 5.2). A straightforward consequence of that observation is that f is a weakly conform *faithful morphism*. Thus, applying Lemma 4.2 and (5.6.1), we have $\rho_2(\mathcal{C}^S) = \rho_2(\mathcal{C}) = \rho$ and, incidentally, $\rho_k(\mathcal{C})\leq \rho_k(\mathcal{C}^S)$.
To multiset S corresponds in X^S a set S' of m distinct points : $S' = \{\{s_i^{(j)}\}\ ;\ 1\leq i\leq p,\ 1\leq j\leq m_i\}$. Apply the restricted partition inequality to X^S: S' admits a k-Tverberg-partition $S'_1,S'_2,\ldots,S'_k$ with partition point σ'. Mapping by f, we obtain a partition of S into k multisets $S_1,S_2,\ldots,S_k$. Since f is a faithful mapping, formula (5.3.1) apply and shows that f(σ') is a k-partition point for S. □

Theorem 10.2 [10]**.** If the restricted partition inequality holds for finite graph-geodetic spaces, then the (unrestricted) partition inequalities hold in every convexity space.

Proof . The general partition problem reduce to finite spaces by Theorem 6.1, then to finite point convex spaces by Proposition 7.1, then to the restricted partition inequality for finite point convex spaces by Proposition 10.1, then to finite interval point-convex convexity spaces by Proposition 8.5. The final reduction to finite graph-geodetic spaces follows from theorem 9.6 : if $\mathcal{C}^*$ denote the graph-geodetic extension of a point-convex interval convexity $\mathcal{C}$, applying successively the restricted partition inequality for $\mathcal{C}^*$, inequality (9.6.1), equality (9.6.2) and the general equality $\dot{\rho}_2 = \rho_2$, we are led, for $\rho_2(\mathcal{C})\geq 3$, to :

$$\dot{\rho}_k(\mathcal{C}) \leq \dot{\rho}_k(\mathcal{C}^*) \leq (k\text{-}1)\, \dot{\rho}_2(\mathcal{C}^*) = (k\text{-}1)\, \rho_2(\mathcal{C}) .$$

Since the partition conjecture is true when $\rho_2(\mathcal{C})\leq 2$ [15], our theorem follows. □

11 Open questions, remarks and suggestions

The results we obtained here tend to give more credit to the "partition conjecture". We feel that similar methods could apply to give a satisfactory answer to the following particular questions.

Conjecture 1 (G. Hahn, N. Sands, R. Woodrow, 1981 [13]) — In a infinite bridged graph G, every finite set of vertices is contained in some finite bridged induced subgraph (let us recall that a graph is *bridged* when it admits no isometric cycle of length at least 4).

Conjecture 2 The k-partition dimension of the geodetic convexity of a connected triangulated graph G is at most $(k\text{-}1)\omega$, where ω is the maximum size of a complete subgraph of G.

Conjecture 3 The k-partition dimension of the minimal-path convexity of a connected graph G (see [8]) is at most $(k\text{-}1)\omega$, where ω is the maximum size of a complete subgraph of G.

Nevertheless, I got the impression that the study of the partition problem in the setting of Graph Theory is not satisfactory. To attack the partition problem, I suggest to study it in the more general setting of finite metrical convexity spaces with Radon-dimension ρ, looking at the interplay between Radon points of minimal Radon-dependent sets one one hand, and *(k-1)-cores* of multisets of size $(k\text{-}1)\rho+1$ on the other hand. See [9] for such a proof of the partition conjecture for rank 3 oriented matroids.

Another direction we intend to explore in the near future is the following

Problem 4. Extend methods and results of the present paper to other dimensional parameters.

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