Cyclic polytopes and oriented matroids*

Raul Cordovil†
Departamento de Matemática
Instituto Superior Técnico
Av. Rovisco Pais - 1096 Lisboa Codex - Portugal
cordovil@math.ist.utl.pt

Pierre Duchet
C.N.R.S., Combinatoire
Université Pierre et Marie Curie - Case 189
4, Place Jussieu - 75005 Paris
duchet@ccr.jussieu.fr

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Abstract

Consider the moment curve in the real Euclidean space $\mathbb{R}^d$ defined parametrically by the map $\gamma : \mathbb{R} \to \mathbb{R}^d$, $t \mapsto \gamma(t) = (t, t^2, \dots, t^d)$. The cyclic d-polytope $\mathcal{C}_d(t_1, \dots, t_n)$ is the convex hull of the n, $n > d$, different points on this curve. The matroidal analogues are the alternating oriented uniform matroids. A polytope [resp. matroid polytope] is called cyclic if its face lattice is isomorphic to that of $\mathcal{C}_d(t_1, \dots, t_n)$. We give combinatorial and geometrical characterizations of cyclic [matroid] polytopes. A simple evenness criterion determining the facets of $\mathcal{C}_d(t_1, \dots, t_n)$ was given by David Gale. We characterize the admissible orderings of the vertices of the cyclic polytope, i.e., those linear orderings of the vertices for which Gale's evenness criterion holds. Proofs give a systematic account on an oriented matroid approach to cyclic polytopes.

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1 Introduction

The *standard d-th cyclic polytope* with n, $n > d$, vertices, denoted by $\boldsymbol{\mathcal{C}}_d(t_1, t_2, \dots, t_n)$, was discovered by Constantin Carathéodory [8, 9] in the context of harmonic analysis, and has been rediscovered many times; it is defined as the convex hull in $\mathbb{R}^d$ of $n,\ n > d$, different points $\gamma(t_1), \dots, \gamma(t_n)$ on the *moment curve* in the real Euclidean space $\mathbb{R}^d$ defined parametrically by the map $\gamma : \mathbb{R} \to \mathbb{R}^d,\ t \mapsto \gamma(t) = (t, t^2, \dots, t^d)$. Moment curves offer a rich spatial structure that occurs in various domains: let us mention, for instance, the use of a d-th moment curve for embedding m-dimensional simplicial complexes in $\mathbb{R}^{2m+1}$, existence of mutually adjacent convex bodies in $\mathbb{R}^3$ [13], or for constructing n "neighborly" convex bodies in $\mathbb{R}^d$ (any two of them share a common facet) [15]. d-th cyclic polytopes are the simplest examples of d-dimensional *neighborly polytopes*, i.e., in which every subset of k, $k \leq \lfloor d/2 \rfloor$, vertices is the vertex set of a face of the polytope. Neighborly d-polytopes play a prominent role in the theory of polytopes since, among d-polytopes with n vertices, they have the greatest number of facets ["Upper bound theorem," conjectured by Theodore S. Motzkin [28] and proved by Peter McMullen [26, 27]]. Recent and quite unexpected other applications of cyclic polytopes may be found in [2, 24, 29, 39].

Let us recall that the set of all the faces [including the improper faces] of a [convex] polytope $\boldsymbol{\mathcal{P}}$, when partially ordered by inclusion, is a finite lattice called the *face lattice of* $\boldsymbol{\mathcal{P}}$. Two polytopes are said to be *combinatorially equivalent*, or of the same *combinatorial type*, if they have isomorphic face lattices. *Cyclic polytopes* are precisely those which are combinatorially equivalent to the standard cyclic polytope $\boldsymbol{\mathcal{C}}_d(t_1, t_2, \dots, t_n)$.

As Branko Grünbaum noticed [22], it is not surprising that many other curves can take the place of the moment curve for developing the theory of cyclic polytopes: examples can be found in [12, 20, 21, 33]. A parameterized curve $\alpha : \mathbb{R} \to \mathbb{R}^d,\ t \mapsto \alpha(t)$ is called a *d-th cyclic curve* [resp. *d-th order curve*] when the convex hull of any n, $n \geq d+1$, different points $\alpha(t_1), \dots, \alpha(t_n)$ is a cyclic d-polytope [resp. no affine hyperplane $\boldsymbol{H}$ in $\mathbb{R}^d$ meets the curve in more than d points]. Answering an implicit question in Grünbaum [22] we prove that a curve is cyclic if and only if it is an order curve. Gale [21], for instance, choose the *trigonometric moment curve* $\alpha : \mathbb{R} \to \mathbb{R}^{2d}$, defined by $\alpha(t) = (\cos t, \sin t, \dots, \cos dt, \sin dt) \in \mathbb{R}^{2d}$. The convex hull of the n, $n \geq 2k+1$, points $\{\alpha(2\pi j/n),\ j = 0, 1, \dots, n-1\}$ is the *regular cyclic* $2d$*-dimensional polytope with* n *vertices*. In this sense, the cyclic polytopes are a satisfactory d-dimensional analogue of a plane convex n-gon.

A more restrict notion of equivalence of polytopes is the concept of "geometrical type". Two polytopes are said to be *geometrically equivalent*, or of the same *geometrical type*, if there is a one-to-one correspondence between vertex sets that preserve not only the affine dependencies of the points, but also preserves the bipartition of the coefficients of any such minimal dependence into positive and negative ones (for details see Section 2). Geometrical equivalence extends to any finite point set in $\mathbb{R}^d$: the corresponding equivalence class describes the "geometry of the set" and is usually called its "geometrical type."

Oriented matroids constitute an unifying and efficient setting for modeling geometrical types and also their interplay with combinatorial types. The present paper adopts this point of view. Investigating the geometry of cyclic polytopes, we review and renovate known results, and obtain new ones. Several tools, introduced in an earlier version of our manuscript (quoted as [10] by various authors), have received, afterwards, more general interest.

The content is as follows. In Section 2 we introduce prerequisites dealing with oriented matroids, polytopes, and their matroidal analogues, i.e., "matroid polytopes." We present in Section 3 the geometry type of the standard cyclic polytope $\boldsymbol{\mathcal{C}}_d(t_1, \ldots, t_n)$, known in the literature as the "alternating oriented uniform matroid of rank $d+1$ on n elements." We give a short proof of an unpublished "folklore," stating that some $\boldsymbol{\mathcal{C}}_d(t_1, t_2, \ldots, t_n)$ appears in every sufficiently large set of points in general position in $\mathbb{R}^d$ (no other polytopal geometry shares this property!). In Section 4, we examine the facial structure of the cyclic polytopes. A simple "evenness criterion" determining the facets [resp. faces] of $\boldsymbol{\mathcal{C}}_d(t_1, t_2, \ldots, t_n)$, was given by David Gale [21] [resp. Geoffrey Shephard]. We give a direct short proof of an extension of Gale's and Shephard's criteria to "cyclic matroid polytopes." Cyclic matroid polytopes are thoroughly investigated in Section 5. All cyclic [matroid] polytopes of even dimension [odd rank] are geometrically equivalent. A short direct proof of this "rigidity property," proved more generally for even dimensional neighborly polytopes by Ido Shemer [34], is given here. For odd dimension [even rank], rigidity fails, in a strong sense: there are polytopes combinatorially equivalent to cyclic polytopes but of a different geometrical type. We describe a possible construction of all these odd dimensional cyclic polytopes. Making use of the some results on "inseparability graphs" of oriented matroids [11, 32], we prove two results that emphasize the very special place of alternating oriented matroids among realizable cyclic matroid polytopes of even rank. To conclude, we provide a characterization of the *admissible orderings*, i.e., the linear orderings of the vertices of a cyclic [matroid] polytope such that Gale's evenness criterion holds.

Although a full understanding of the text supposes the reader to be familiar with matroid theory [40, 41] and oriented matroid theory [3]. Additional information concerning polytopes may be found in [1, 7, 19, 26, 30, 35, 37, 38, 42, 43] and their references. Nevertheless the paper is essentially self-contained and may support an introductory course on oriented matroids and/or polytopes.

2 Oriented Matroids and Polytopes

We remember that a finite subset X, $X \subset \mathbb{R}^d$ is *affinely dependent* if there are reals λ_x, $x \in X$ in which some $\lambda_x \neq 0$ and $\sum_{x \in X} \lambda_x = 0$ satisfying $\sum_{x \in X} x \cdot \lambda_x = 0$. Let S be a fixed finite set of points in $\mathbb{R}^d \setminus \{0\}$. Let $\mathfrak{C}$ be the set of the minimal [to inclusion] affine dependent subset of S. The pair $(S, \mathfrak{C})$ is called the *matroid of the affine dependencies on the ground set* S, and denoted $\underline{\mathbb{A}_{\mathit{ff}}}(S)$. We say that $\mathfrak{C}$ is the set of the *circuits* of the matroid $\underline{\mathbb{A}_{\mathit{ff}}}(S)$. The natural ordering of $\mathbb{R}$ induces a canonical *circuit signature* of the circuits of $\mathfrak{C}$. Observe that if $\mathcal{C} \in \mathfrak{C}$ then the map $\lambda : \mathcal{C} \to \mathbb{R} \setminus \{0\}$, $x \mapsto \lambda_x$, is unique up to multiplication by a non-zero real number. Thus $(\mathcal{C}, \lambda)$ determines a pair of opposite signed sets:

$$\boldsymbol{\mathcal{C}} := (\boldsymbol{\mathcal{C}}^+ = \{x \in \mathcal{C} : \lambda_x > 0\};\ \boldsymbol{\mathcal{C}}^- = \{x \in \mathcal{C} : \lambda_x < 0\})$$

$$\text{and} \quad -\boldsymbol{\mathcal{C}} := ((-\boldsymbol{\mathcal{C}})^+ = \boldsymbol{\mathcal{C}}^-;\ \ (-\boldsymbol{\mathcal{C}})^- = \boldsymbol{\mathcal{C}}^+).$$

We say that $\{\boldsymbol{\mathcal{C}}, -\boldsymbol{\mathcal{C}}\}$ is a pair of opposite *signed circuits* determined by the finite set S. The underlying set $\underline{\boldsymbol{\mathcal{C}}} := \boldsymbol{\mathcal{C}}^+ \cup \boldsymbol{\mathcal{C}}^- [= \mathcal{C}]$ is called the *support* of the opposite signed circuits $\boldsymbol{\mathcal{C}}$, $-\boldsymbol{\mathcal{C}}$. Let $\boldsymbol{\mathfrak{C}}$ be the collection of signed circuits determined by the set of circuits $\mathfrak{C}$ of $\underline{\mathbb{A}_{\mathit{ff}}}(S)$. The pair $(S, \boldsymbol{\mathfrak{C}})$ is called the *oriented matroid of the affine dependencies on the ground set* S and denote $\mathbb{A}_{\mathit{ff}}(S)$. Note that the elements of $\boldsymbol{\mathfrak{C}}$ satisfy the following *signed elimination property*:

- For all $\mathcal{C}_1, \mathcal{C}_2 \in \boldsymbol{\mathfrak{C}}$, and $a \in \mathcal{C}_1^+ \cap \mathcal{C}_2^-$ there is a signed circuit $\mathcal{C}_3 \in \boldsymbol{\mathfrak{C}}$ such that $\mathcal{C}_3^+ \subset \mathcal{C}_1^+ \cup \mathcal{C}_2^+ \setminus \{a\}$ and $\mathcal{C}_3^- \subset \mathcal{C}_1^- \cup \mathcal{C}_2^- \setminus \{a\}$.

By forgetting the signs we recover the *underlying matroid* $\underline{\mathbb{A}_{\mathit{ff}}}(S)$. Consider a pair $\{A, B\}$, $A \cap B = \emptyset$, $A \cup B \subset S$, where S is a finite subset of $\mathbb{R}^d$. The pair $\{A, B\}$ is called a *Radon partition in* S provided $\text{conv}(A) \cap \text{conv}(B) \neq \emptyset$. We say that $\{A, B\}$ is a *primitive Radon partition in* S if it is minimal in the sense that it does not extend any other Radon partition in S, see [23]. Then it is clear that the pair $\{A, B\}$ is a primitive Radon partition in S if and only

$(\boldsymbol{\mathcal{C}}^+ = A\,, \boldsymbol{\mathcal{C}}^- = B)$ is a signed circuit of $\mathbb{A}_{ff}(S)$. We observe that the pair $\{A, B\}$ is a *non-Radon partition in* S (i.e., $\operatorname{conv}(A) \cap \operatorname{conv}(B) = \emptyset$) if and only if there is an affine hyperplane $\boldsymbol{H}$ in $\mathbb{R}^d$ with corresponding halfspaces $\boldsymbol{H}^+$ and $\boldsymbol{H}^-$ such that $A \subset \boldsymbol{H}^+$, $B \subset \boldsymbol{H}^-$ and $(A \cup B) \cap \boldsymbol{H} = \emptyset$. The reader is referred to [14, 16, 17] for a discussion of Radon partitions. Note that the set of the "different Radon types determined by a set of n points in $\mathbb{R}^d$" studied by Jürgen Eckhoff [16, 17], coincides with the set of "non isomorphic oriented matroids of the affine dependencies determined by a set of n points in $\mathbb{R}^d$ ".

The cardinal of the maximal affinely independent subsets of Y, $Y \subset S$, is said to be the *rank* of Y in $\mathbb{A}_{ff}(S)$. Set $\operatorname{rank}(\mathbb{A}_{ff}(S)) = \operatorname{rank}(\underline{\mathbb{A}_{ff}}(S)) = \operatorname{rank}(S)$. A subset F, $F \subset S$, is said a *flat* of $\mathbb{A}_{ff}(S)$ [or $\underline{\mathbb{A}_{ff}}(S)$] if $\operatorname{rank}(F) < \operatorname{rank}(F \cup \{s\})$ for all s, $s \in S \backslash F$. The flats of rank 1,2, and corank 1 are called *points, lines* and *hyperplanes* of the matroid $\mathbb{A}_{ff}(S)$, respectively. Suppose $\operatorname{rank}(S) = d+1$. [This restriction is not a handicap; we can reduce the general case to this case.] The affine subspace $\boldsymbol{H} = \langle H \rangle$ of $\mathbb{R}^d$ generated by the elements of the hyperplane H of the matroid $\mathbb{A}_{ff}(S)$ is an affine hyperplane of $\mathbb{R}^d$. For every affine hyperplane $\boldsymbol{H}$ of this type we choose a positive $\boldsymbol{H}^+$ and and negative $\boldsymbol{H}^-$ halfspaces. Thus, the hyperplane H of $\mathbb{A}_{ff}(S)$ determines a pair of opposite signed sets:

$$\boldsymbol{\mathcal{C}}^* := (\, (\boldsymbol{\mathcal{C}}^*)^+ = \{x :\; x \in \{S \backslash H\} \cap \boldsymbol{H}^+\};\;\; (\boldsymbol{\mathcal{C}}^*)^- = \{x :\; x \in \{S \backslash H\} \cap \boldsymbol{H}^-\}\,)$$

$$\text{and} \quad -\boldsymbol{\mathcal{C}}^* := ((-\boldsymbol{\mathcal{C}}^*)^+ = (\boldsymbol{\mathcal{C}}^*)^-;\;\; (\boldsymbol{\mathcal{C}}^*)^- = (\boldsymbol{\mathcal{C}}^*)^+).$$

We say that $\{\boldsymbol{\mathcal{C}}^*, -\boldsymbol{\mathcal{C}}^*\}$ is a pair of opposite *signed cocircuits* determined by the set S, $S \subset \mathbb{R}^d \setminus \{0\}$. The underlying set $\underline{\boldsymbol{\mathcal{C}}}^* = \boldsymbol{\mathcal{C}}^+ \cup \boldsymbol{\mathcal{C}}^- = S \setminus H$ is called the *support* of the opposite signed cocircuits $\boldsymbol{\mathcal{C}}^*$, $-\boldsymbol{\mathcal{C}}^*$. Let $\mathfrak{C}^*$ be the collection of all signed cocircuits determined by the point set S. The oriented matroid $\mathbb{A}_{ff}(S)$ can be also encoded by the pair $(S, \mathfrak{C}^*)$. Indeed the sets $\mathfrak{C}$ and $\mathfrak{C}^*$ satisfy the *orthogonality property*:

$(\ast)$ For all $\boldsymbol{\mathcal{C}} \in \mathfrak{C}$ and $\boldsymbol{\mathcal{C}}^* \in \mathfrak{C}^*$ such that $|\underline{\boldsymbol{\mathcal{C}}} \cap \underline{\boldsymbol{\mathcal{C}}}^*| \geq 2$, both the sets $\{\boldsymbol{\mathcal{C}}^+ \cap (\boldsymbol{\mathcal{C}}^*)^+\} \cup \{\boldsymbol{\mathcal{C}}^- \cap (\boldsymbol{\mathcal{C}}^*)^-\}$ and $\{\boldsymbol{\mathcal{C}}^+ \cap (\boldsymbol{\mathcal{C}}^*)^-\} \cup \{\boldsymbol{\mathcal{C}}^- \cap (\boldsymbol{\mathcal{C}}^*)^+\}$ are non-empty.

$\mathfrak{C}$ [resp. $\mathfrak{C}^*$] is the set of minimal signed subsets of S satisfying $(\ast)$ relatively to the set $\mathfrak{C}^*$ [resp. $\mathfrak{C}$], and $(\mathfrak{C}^*)^* = \mathfrak{C}$. The pair $(S, \mathfrak{C}^*)$ is also a new rank $(n-r)$ oriented matroid called the *orthogonal* or *dual* of $\mathbb{A}_{ff}(S)$ and denoted $(\mathbb{A}_{ff}(S))^*$ or $\mathbb{A}^*_{ff}(S)$. The *restriction* of $\mathbb{A}_{ff}(S)$ to the ground set $S \setminus \{s\}$ is denoted by $\mathbb{A}_{ff}(S) \setminus s$. Set $\mathbb{A}_{ff}(S)/s := (\mathbb{A}^*_{ff}(S) \setminus s)^*$. We say that $\mathbb{A}_{\mathit{ff}}(S)/s$ is the *contraction* of $\mathbb{A}_{\mathit{ff}}(S)$ by the element s. Note that $(\mathbb{A}_{\mathit{ff}}(S) \setminus s) \setminus s' = (\mathbb{A}_{\mathit{ff}}(S) \setminus s') \setminus s$ and for all $s, s' \in S$.

The notion of "oriented matroid" can be axiomatized (see [3, Definition 3.2.1]). Thus, we obtain a general class, where there are many oriented matroids non-realizable, i.e., non-isomorphic to matroids of type $\mathbb{A}_{\mathit{ff}}(S)$, see [3]. For many combinatorial purposes the class of all oriented matroids can be seen as an useful completion of the class of realizable oriented matroids. In particular the class of all oriented matroids is closed for the important operation of "local perturbation" (see [3] for details).

A [convex] *d'-polytope* is the ordinary convex hull of a finite subset of $\mathbb{R}^d$ whose affine dimension is d'. The matroidal analogue notion is the concept of "matroid polytope." Consider an *acyclic oriented matroid* $\mathcal{M}$, i.e., suppose that all its signed circuits have positive and negative elements. A *facet* of $\mathcal{M}$ is a hyperplane H [of $\mathcal{M}$] such that $S(\mathcal{M}) \setminus H$ supports a *positive cocircuit* of $\mathcal{M}$, i.e., a cocircuit whose negative part is empty. A *face* is an intersection of facets, i.e., a subset F of S such that $S \setminus F$ is a union of [the support of] positive cocircuits of $\mathcal{M}$. The collection of all faces of $\mathcal{M}$ ordered by inclusion is a finite lattice, $\mathfrak{Fac}(\mathcal{M})$, called the (*Las Vergnas*) *face lattice* of $\mathcal{M}$. Note that $\text{rank}(\mathcal{M}) = \text{rank}(\mathfrak{Fac}(\mathcal{M}))$, and the face lattice $\mathfrak{Fac}(\mathcal{M})$ has two improper faces: $\emptyset$ and [the ground set] $S(\mathcal{M})$.

Let $\boldsymbol{\mathcal{P}}$ be a d-polytope of dimension in $\mathbb{R}^d$, and let $V := \text{vert}(\boldsymbol{\mathcal{P}})$ be its vertex set. Consider the [realizable] matroid polytope of the affine dependencies of its vertex set $\boldsymbol{\mathbb{A}_{\mathit{ff}}}(V)$. Note that there is a natural isomorphism $\phi : \mathfrak{Fac}(\boldsymbol{\mathcal{P}}) \to \mathfrak{Fac}(\boldsymbol{\mathbb{A}_{\mathit{ff}}}(V))$, $F \mapsto \phi(F) := \text{vert}(F)$. For this reason, an acyclic oriented matroid $\mathcal{M}$ such that all the elements of $S(\mathcal{M})$ are vertices (i.e., rank 1 faces) of the face lattice $\mathfrak{Fac}(\mathcal{M})$ will be called a *matroid polytope*. The oriented matroids that are also matroid polytopes will be denoted by a bold symbol.

Let S,S' be two finite subsets of $\mathbb{R}^d$. We say that S and S' are *geometrically equivalent*, or have the same *geometrical type*, if the oriented matroids $\mathbb{A}_{\mathit{ff}}(S)$ and $\mathbb{A}_{\mathit{ff}}(S')$ are isomorphic. We say that two matroid polytopes $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{M}}'$ are *combinatorially equivalent*, or have the same *combinatorial type*, if the corresponding face lattices are isomorphic. Suppose that the "reorientation on A of [all the signed circuits of] $\boldsymbol{\mathcal{M}}$," denoted by ${}_{\overline{A}}\boldsymbol{\mathcal{M}}$, is also a matroid polytope. The partition $\{A, S \setminus A\}$ of the ground set $S(\boldsymbol{\mathcal{M}})$ is called a *non-Radon partition* [relative to $\boldsymbol{\mathcal{M}}$] of the ground set S. [The "reorientation on A" of the signed circuit $\boldsymbol{\mathcal{C}}$ is the signed set ${}_{\overline{A}}\boldsymbol{\mathcal{C}} := (({}_{\overline{A}}\boldsymbol{\mathcal{C}})^+ = \{\boldsymbol{\mathcal{C}}^+ \setminus A\} \cup \{A \cap \boldsymbol{\mathcal{C}}^-\};\ ({}_{\overline{A}}\boldsymbol{\mathcal{C}})^- = \{\boldsymbol{\mathcal{C}}^- \setminus A\} \cup \{A \cap \boldsymbol{\mathcal{C}}^+\})$. The concept of "reorientation" is the matroidal analogue of the notion of "nonsingular projective permissible transformation," for details see for example [22]. Note that we have ${}_{\overline{A}}\mathcal{M} = {}_{\overline{S\setminus A}}\mathcal{M}$, for every oriented matroid $\mathcal{M} = (S, \mathfrak{C})$.] We observe that if $\boldsymbol{\mathcal{M}} = \mathbb{A}_{\boldsymbol{ff}}(S)$ then the two notions of non-Radon partition here defined, coincide. With the language of primitive Radon partitions, Marilyn Breen [5] has observed:

Proposition 2.1 ([5]). *The combinatorial type of a polytope is determined by its geometrical type.* $\square$

The next results extend to matroid polytopes similar theorems of [5].

Proposition 2.2 ([25]). *Consider a matroid polytope $\boldsymbol{\mathcal{M}} = (S, \mathfrak{C})$ and let F be a subset of the ground set S. Then the following two assertions are equivalent:*

- *F is a face of $\boldsymbol{\mathcal{M}}$.*
- *For all signed circuits $\boldsymbol{\mathcal{C}} \in \mathfrak{C}$, $\boldsymbol{\mathcal{C}}^+ \subset F \Longrightarrow \boldsymbol{\mathcal{C}}^- \subset F$.* $\square$

A rank r matroid $\mathcal{M}$ is called *uniform* if its independent sets are exactly those subsets of the ground set $S(\mathcal{M})$ whose cardinal is at most r. A matroid polytope $\boldsymbol{\mathcal{M}}$ is called *simplicial* if every of its faces F, $F \neq S(\boldsymbol{\mathcal{M}})$, is an independent set. From Proposition 2.2 we conclude:

Corollary 2.3. *Let $\boldsymbol{\mathcal{M}}$ be a simplicial matroid polytope. Let F be a proper subset of the ground set $S(\boldsymbol{\mathcal{M}})$. The following three assertions are equivalent:*

- *F is a face of $\boldsymbol{\mathcal{M}}$.*
- *$\{A, S \setminus A\}$ is a non-Radon partition of S, for every $A \subset F$.*
- *$\boldsymbol{\mathcal{C}}^+ \not\subset F$, for every signed circuit $\boldsymbol{\mathcal{C}}$ of $\boldsymbol{\mathcal{M}}$.*

Even for uniform matroid polytopes combinatorial equivalence does not imply geometrical equivalence. $\square$

3 Alternating orientations and d-th cyclic curves

Set $[n] := \{1, 2, \ldots, n\}$ for every $n \in \mathbb{N}_+$, and set $[0] := \emptyset$. When necessary we consider also $[n]$ as the linear ordered set $\{1 < 2 < \cdots < n\}$. Let $\underline{\mathbb{U}}_r([n])$ be the rank r uniform matroid on the ground set $[n]$. It is well known [4, Corollary 3.9.1] that an *alternating orientation* $\mathfrak{C}$ of the circuits of $\underline{\mathbb{U}}_r(n)$ can be associated with the linear ordered set $[n]$:

- Every circuit $\mathcal{C} = \{i_1 < \cdots < i_{r+1}\}$ of $\underline{\mathbb{U}}_r(n)$ is the support of a pair of opposite signed circuits, $\boldsymbol{\mathcal{C}}, -\boldsymbol{\mathcal{C}}$ of $\mathfrak{C}$, determined by the signature $\mathrm{sg}_{\boldsymbol{\mathcal{C}}}(i_{j+1}) = -\mathrm{sg}_{\boldsymbol{\mathcal{C}}}(i_j),\ j \in [r]$.

The oriented matroid so obtained, $\mathbb{A}_r([n]) := ([n], \mathfrak{C})$, is called the *alternating oriented [uniform] matroid* of rank r on the linear ordered set $[n]$. Notice that $\mathbb{A}_r([n])$ is a matroid polytope if $r \geq 3$. Properties of alternating oriented matroid have a nice simple translation in terms of basis orientation. A *basis orientation* of an oriented matroid $\mathcal{M} = (S, \mathfrak{C})$ is a mapping χ of the set of ordered bases of $\mathcal{M}$ to $\{-1, 1\}$ satisfying the following two properties:

- χ is alternating.
- For any two linear ordered bases $B_1 = \{a \prec s_1 \prec \cdots \prec s_r\}$ and $B_2 = \{b \prec s_1 \prec' \cdots \prec' s_r\}$, $a \neq b$, of $\mathcal{M}$, we have $\chi(B_1) = -\mathrm{sg}_{\boldsymbol{\mathcal{C}}}(a)\mathrm{sg}_{\boldsymbol{\mathcal{C}}}(b)\chi(B_2)$, where $\boldsymbol{\mathcal{C}}$ denotes one of the two opposite signed circuits of $\mathcal{M}$ such that $a, b \in \underline{\boldsymbol{\mathcal{C}}} \subset B_1 \cup \{b\} = B_2 \cup \{a\}$.

If $\chi(B) = 1$ we say that B is a *positive* basis of $\mathcal{M}$. Note that the map χ' of the linear ordered bases of $\mathbb{A}_{ff}(S)$, $B \mapsto \chi'(B) = \mathrm{sign}\,(\det(B))$ is a basis orientation of $\mathbb{A}_{ff}(S)$.

Proposition 3.1 ([4]). *A rank $r, r \geq 1$, uniform oriented matroid $\mathbb{U}_r(S)$ is an alternating oriented matroid if and only if, for some linear ordering $S_\prec$ of the ground set S, every ordered basis $B_\prec, B_\prec \subset S_\prec$, is positive.* $\square$

The geometrical type of the standard cyclic polytopes are the alternating oriented matroid polytopes. More precisely we have:

Theorem 3.2 ([4, 6]). *Consider the points $\gamma(t_1), \ldots, \gamma(t_n)$, $t_1 < \cdots < t_n$, $n > d > 2$, of the moment curve $\gamma : \mathbb{R} \to \mathbb{R}^d$, $t \mapsto \gamma(t) = (t, t^2, \ldots, t^d)$. Then*

$$\boldsymbol{\mathbb{A}_{ff}}(\{\gamma(t_1)), \gamma(t_2), \ldots, \gamma(t_n)\}) = \boldsymbol{\mathbb{A}}_{d+1}(\{\gamma(t_1) \prec \gamma(t_2) \prec \cdots \prec \gamma(t_n)\}).$$

Proof. Breen's proof [6] uses Gale's evenness criterion (see Proposition 4.2 below). In fact, as suggested in [4], an effective simple calculus using Vandermonde's determinants suffices: from the definitions we know that the matroid $\boldsymbol{\mathbb{A}_{ff}}(\{\gamma(t_1), \gamma(t_2), \ldots, \gamma(t_n)\})$ is a uniform matroid of rank $d+1$. Let $\mathcal{C} = \{\gamma(t_{i_1}), \ldots, \gamma(t_{i_{d+2}})\}$, $t_{i_1} < \cdots < t_{i_{d+2}}$, be the support of one of its signed circuits $\boldsymbol{\mathcal{C}}$. The calculus of the coefficients $\lambda_1, \ldots, \lambda_{d+2}$ of an affine combination,

$$\sum_{j=1}^{j=d+2} \lambda_j \gamma(t_{i_j}) = 0 \ \text{ with } \ \sum_{j=1}^{j=d+2} \lambda_j = 0,$$

shows that λ_{i+1} and λ_i have opposite signs for every $i \in [d+1]$. Hence the conclusion follows. $\square$

Lemma 3.3 (Signature Lemma). *Let $\mathfrak{C}$ [resp. $\mathfrak{C}^*$] be the set of signed circuits [resp. cocircuits] of an oriented uniform matroid $\mathbb{U}_r([n])$. Suppose that one the following two conditions is verified:*

- *If $\boldsymbol{\mathcal{C}} \in \mathfrak{C}$ and $i, i+1 \in \underline{\boldsymbol{\mathcal{C}}}$, then $\mathrm{sg}_{\boldsymbol{\mathcal{C}}}(i+1) = -\mathrm{sg}_{\boldsymbol{\mathcal{C}}}(i)$, for all $i \in [n-1]$.*
- *If $\boldsymbol{\mathcal{C}}^* \in \mathfrak{C}^*$, and $i, i+1 \in \underline{\boldsymbol{\mathcal{C}}^*}$, then $\mathrm{sg}_{\boldsymbol{\mathcal{C}}^*}(i+1) = \mathrm{sg}_{\boldsymbol{\mathcal{C}}^*}(i)$, for all $i \in [n-1]$.*

Then $\mathbb{U}_r([n])$ is the alternating oriented matroid $\mathbb{A}_r([n])$.

Proof. It is well known that given any circuit $\mathcal{C}$ of a matroid $\mathcal{M} = (S, \mathfrak{C})$ and two elements a, b, $a \neq b$, $a, b \in \mathcal{C}$, there is a cocircuit $\mathcal{C}^*$ of $\mathcal{M}$ such that $\mathcal{C} \cap \mathcal{C}^* = \{a, b\}$, see for example [3, Lemma 3.4.2]. Thus, by the orthogonality property $(\ast)$, the assertions relative to circuits and to cocircuits are equivalent. We establish the lemma for circuits, proving that, for every signed circuit $\boldsymbol{\mathcal{C}}$ with support $\underline{\boldsymbol{\mathcal{C}}} = \{i_1, \dots, i_{r+1}\}$, $i_1 < \cdots < i_{r+1}$, and for every pair of natural numbers p, q, $1 \le p < q \le r+1$, we have

$$\mathrm{sg}_{\boldsymbol{\mathcal{C}}}(i_q) = (-1)^{q-p}\mathrm{sg}_{\boldsymbol{\mathcal{C}}}(i_p). \tag{3.3.1}$$

We use induction on $i_q - i_p$. If $i_q = i_p + 1$, equation (3.3.1) is the hypothesis. Suppose $i_q - i_p > 1$ and that (3.3.1) is true for all the integers $i_{p'}$ and $i_{q'}$ such that $1 \le i_{q'} - i_{p'} < i_q - i_p$. If $p+1 < q$, there is $s_{i_{p+1}} \in \underline{\boldsymbol{\mathcal{C}}}$ such that $i_p < i_{p+1} < i_q$; then the result follows by the induction hypothesis. Now suppose $q = p+1$. Let j be some element of $S \setminus \underline{\boldsymbol{\mathcal{C}}}$ with $i_p < j < i_{p+1}$. Let $\boldsymbol{\mathcal{C}}'$ [resp. $\boldsymbol{\mathcal{C}}''$] be the signed circuit of $\mathbb{U}_r([n])$ supported by $\{\underline{\boldsymbol{\mathcal{C}}} \setminus \{i_p\}\} \cup \{j\}$ [resp. $\{\underline{\boldsymbol{\mathcal{C}}''} \setminus \{i_{p+1}\}\} \cup \{j\}$]. By induction hypothesis $\mathrm{sg}_{\boldsymbol{\mathcal{C}}'}(i_{p+1}) = -\mathrm{sg}_{\boldsymbol{\mathcal{C}}'}(j)$ and $\mathrm{sg}_{\boldsymbol{\mathcal{C}}''}(j) = -\mathrm{sg}_{\boldsymbol{\mathcal{C}}''}(i_p)$. Hence, by the signed elimination property, $\mathrm{sg}_{\boldsymbol{\mathcal{C}}}(i_{p+1}) = -\,\mathrm{sg}_{\boldsymbol{\mathcal{C}}}(i_p)$. $\square$

Let us give two consequences of Signature Lemma 3.3.

Corollary 3.4 ([4]). *Let $\mathfrak{C}^*$ be the set of signed cocircuits of the rank r alternating oriented matroid $\mathbb{A}_r([n])$. Set $E := \{i \in [n] : i \text{ even}\}$. Then $\overline{_E}\mathfrak{C}^* := \{\overline{_E}\boldsymbol{\mathcal{C}}^* : \boldsymbol{\mathcal{C}}^* \in \mathfrak{C}^*\}$ is the set of the signed circuits of the rank $(n-r)$ alternating oriented matroid $\mathbb{A}_{n-r}([n])$.*

Proof. Let $\boldsymbol{\mathcal{C}}^*$ be a signed cocircuit of $\mathbb{A}_r([n])$. Set $\underline{\boldsymbol{\mathcal{C}}^*} = \{i_1, \dots, i_{n-r+1}\}$, $i_1 < \cdots < i_{n-r+1}$. Applying (3.3.1) and the orthogonality property $(\ast)$ we obtain, for every pair of integers p, q, $1 \le p < q \le n-r+1$:

(3.4.1) $\quad \mathrm{sg}_{\mathcal{C}^*}(i_q) = (-1)^{(q-p)+(i_q-i_p)}\mathrm{sg}_{\mathcal{C}^*}(i_p).$

We have $(-1)^{i_q-i_p} = -1$ if and only if exactly one of the numbers i_p, i_q is even. Hence Corollary 3.4 follows. $\square$

Corollary 3.5. *Let $\boldsymbol{\mathcal{P}}$ be a d-polytope in $\mathbb{R}^d$, $d \geq 2$. Then $\boldsymbol{\mathcal{P}}$ is geometrically equivalent to the alternating oriented matroid [polytope] $\mathbb{A}_{d+1}([n])$ if and only if its vertex set V fulfills both the conditions:*

(3.5.1) *No affine hyperplane $\boldsymbol{H}$ in $\mathbb{R}^d$ meets V in more than d points.*

(3.5.2) *There is a linear ordering $V_{\prec} := \{v_{i_1} \prec v_{i_1} \prec \cdots \prec v_{i_n}\}$ of the elements of V such that no affine hyperplane $\boldsymbol{H}$ in $\mathbb{R}^d$, affine hull of d points of V, separates strictly v_{i_k} from $v_{i_{k+1}}$, for every $k \in [n-1]$.*

Proof. The conditions are trivially necessary. We prove that they are sufficient. From (3.5.1) we conclude that $\mathbb{A}_{\boldsymbol{f\!f}}(V)$ is a matroid uniform of rank $d+1$. Pick $k \in [n-1]$. Let $\boldsymbol{\mathcal{C}}^*$ be a signed cocircuit of $\mathbb{A}_{\boldsymbol{f\!f}}(V)$. Then $V \setminus \underline{\boldsymbol{\mathcal{C}}^*}$ is a set of d elements. From (3.5.2) we know that $\mathrm{sg}_{\mathcal{C}^*}(v_{i_k}) = \mathrm{sg}_{\mathcal{C}^*}(v_{i_{k+1}})$, if $v_{i_k}, v_{i_{k+1}} \in \underline{\boldsymbol{\mathcal{C}}^*}$. From Lemma 3.3 we conclude the isomorphism of oriented matroids $\mathbb{A}_{\boldsymbol{f\!f}}(V) \cong \mathbb{A}_{d+1}([n])$, $v_i \mapsto i$, $i \in [n]$. $\square$

The next Proposition answers a question implicitly raised in [22].

Proposition 3.6. *Let $\alpha : \mathbb{R} \to \mathbb{R}^d$, $t \mapsto \alpha(t) = (\alpha_1(t), \ldots, \alpha_d(t))$ be a parameterized curve. Then α is a d-th cyclic curve if and only it is a d-th order curve.*

Proof. If α is a d-th cyclic curve then it is clear that it is also a d-th order curve. To prove the converse assertion, let us consider an arbitrary set of n, $n \geq d+1$, points $\alpha(t_1), \ldots, \alpha(t_n)$ on the d-th order curve α. We suppose $t_1 < t_2 < \cdots < t_n$. No affine hull of d of those points can separate strictly two points of the form $\alpha(t_k)$ and $\alpha(t_{k+1})$ for any $k \in [n-1]$; otherwise such hyperplane would contain $\alpha(t)$ for some $t \in \mathbb{R}$, $t_k < t < t_{k+1}$. Then $\mathbb{A}_{\boldsymbol{f\!f}}(\{\alpha_1(t), \ldots, \alpha_d(t)\}) = \mathbb{A}_{d+1}(\{\alpha_1(t) \prec \cdots \prec \alpha_d(t)\})$ is a cyclic matroid polytope, by Theorem 3.2 and Corollary 3.5. So α is also a cyclic curve, as required. $\square$

The following "existence theorem" of Bernd Sturmfels, is closely related with ours results and must be mentioned.

Theorem 3.7 ([37]). *Let V be a finite subset of $\mathbb{R}^d$, $d \geq 2$, of at least $d+1$ points, and suppose that there is a linear ordering $V_\prec$ such that $\mathbb{A}_{d+1}(V_\prec)$ is an alternating oriented matroid polytope. Then V is on a d-th order curve.* $\square$

The next result, a generalization to $\mathbb{R}^d$ of the classical "Ramsey type theorem" of Erdős and Szekeres [18]; see Grünbaum [22, Exercise 7.3.6] and Duchet and Roudneff [15] (cf. [3, Proposition 9.4.7]).

Corollary 3.8 ([15]). *For every natural number d, $d \geq 2$, there is a natural number $N(d)$ that every set of $N(d)$ points in general position in $\mathbb{R}^d$ contains the vertices of a cyclic polytope of dimension d. Moreover, the cyclic polytopes are the unique geometrical class of polytopes with this property.*

Proof. Choose a linear ordered set of points $N_\prec$, in general position in $\mathbb{R}^d$ and fix an orientation of $\mathbb{R}^d$. A linear ordered $(d+1)$-subset $S_\prec$, $S_\prec \subset N_\prec$, is colored blue or red depending on the sign of $S_\prec$, viewed as an oriented d-simplex in $\mathbb{R}^d$. If N is large enough, then $N_\prec$ contains a linear ordered n-subset $V_\prec$ whose linear ordered $(d+1)$-subsets are colored the same, by Ramsey theorem. From Proposition 3.1 and Theorem 3.2 we conclude that $\mathbb{A}_{d+1}(V_\prec) = \mathbb{A}_{\mathbf{ff}}(V)$ is a cyclic polytope. This proves the first assertion. The last assertion follows from the observation that all subpolytopes of cyclic polytopes are also cyclic polytopes. $\square$

Obviously, every submatroid of an alternating oriented matroid is again an alternating oriented matroid with respect to the induced linear order. More generally, passing to matroid minors can be handled as follows.

Corollary 3.9 ([4]). *For every pair of natural numbers $\{n, r\}$, $n > r \geq 1$, and every $i \in [n]$, we have the equalities:*

$$\mathbb{A}_r([n]) \setminus i = \mathbb{A}_r(\{1 < 2 \cdots < \widehat{i} < \cdots < n\}) \quad \text{and} \tag{3.9.1}$$

$$\begin{aligned} \mathbb{A}_r([n]) \,/\, i &= {}_{\overline{[i-1]}}\mathbb{A}_{r-1}(\{1 < 2 \cdots < \widehat{i} < \cdots < n\}) = \\ &= {}_{\overline{\{j : i < j \leq n\}}}\mathbb{A}_{r-1}(\{1 < 2 \cdots < \widehat{i} < \cdots < n\}). \end{aligned} \tag{3.9.2}$$

Proof. The Equality (3.9.1) is clear. Set $E := \{j \in [n] : j \text{ even}\}$ and $E' := \{j \in [i-1] : j \text{ even}\} \bigcup \{j : j \text{ odd}, i < j \leq n\}$. By the orthogonality property ($\star$), and using Corollary 3.4 we have

$$(\mathbb{A}_r([n]) \,/\, i)^* = \mathbb{A}_r^*([n]) \setminus i = {}_{\overline{E}}\mathbb{A}_{n-r}(\{1 < \cdots < \widehat{i} < \cdots < n\}).$$

Applying Corollary 3.4 again, we obtain

$$\mathbb{A}_r([n]) \,/\, i = ((\mathbb{A}_r([n]) \,/\, i)^*)^* = {}_{\overline{E'}}\,({}_{\overline{E}}\,\mathbb{A}_{r-1}(\{1 < \cdots < \widehat{i} < \cdots < n\})),$$

which is the Equality (3.9.2). $\square$

4 Cyclic matroid polytopes: facial structure

In order to describe in a matroidal way the facial structure of $\boldsymbol{\mathcal{C}}_d(t_1, \ldots, t_n)$, we introduce some terminology. The matroid polytopes combinatorially equivalent to the alternating oriented matroids of rank r, $r \geq 3$, are called *cyclic matroid polytopes.* We will denote $\boldsymbol{\mathcal{C}m}_r(S)$ a rank r cyclic matroid polytope on the ground set S. We remember that a linear ordering $S_\prec := \{s_1 \prec s_2 \prec \cdots \prec s_n\}$ of the elements of S is an admissible ordering for $\boldsymbol{\mathcal{C}m}_r(S)$, if $\mathfrak{Fac}(\boldsymbol{\mathcal{C}m}_r(S)) = \mathfrak{Fac}(\mathbb{A}_r(\{s_1 \prec s_2 \prec \cdots \prec s_n\}))$. In the following we suppose that the natural ordering $\{1 < 2 < \cdots < n\}$ is an admissible order for the matroid polytope $\boldsymbol{\mathcal{C}m}_r([n])$. Generalizing similar concepts for polytopes, we say that a matroid polytope $\boldsymbol{\mathcal{M}}$ is said to be $k-$*neighborly* if every subset of k points of the ground set $S(\boldsymbol{\mathcal{M}})$ is a proper face of $\boldsymbol{\mathcal{M}}$. A rank r matroid polytope $\boldsymbol{\mathcal{M}}$ is called *neighborly* if it is $\lfloor (r-1)/2 \rfloor-$neighborly. The proof of the following Proposition easily follows from the definitions and Proposition 2.2, and it is omitted.

Proposition 4.1. *A matroid polytope $\boldsymbol{\mathcal{M}}$ is k-neighborly if and only if $|\boldsymbol{\mathcal{C}}^+| \geq k+1$ for all signed circuits $\boldsymbol{\mathcal{C}}$ of $\boldsymbol{\mathcal{M}}$. A rank $2k+1$ matroid polytope $\boldsymbol{\mathcal{M}}$ is neighborly if and only if $\mathcal{M}$ is uniform and $|\boldsymbol{\mathcal{C}}^+| = |\boldsymbol{\mathcal{C}}^-| = k+1$ for every signed circuit $\boldsymbol{\mathcal{C}}$ of $\boldsymbol{\mathcal{M}}$.* □

We present a matroidal version of Gale's evenness criterion:

Proposition 4.2 (Gale's evenness criterion for facets, [21]). *Let n and r be two integers with $n > r > 2$. Then a $(r-1)-$subset F, $F \subset [n]$, is a facet of the alternating matroid polytope $\mathbb{A}_r([n])$ if and only if every two elements of $[n] \setminus F$ are separated on $[n]$ by an even number of elements of F.*

Proof. Put $[n] \setminus F = \{i_1 < \cdots < i_p \cdots < i_q < \cdots < i_{n-r+1}\}$. The number of elements of F between i_p and i_q is $i_q - i_p - (q-p)$. F is a facet of $\mathbb{A}_r([n])$ if and only if $[n] \setminus F$ supports a positive cocircuit $\boldsymbol{\mathcal{C}}^*$. From Equation (3.4.1) we know that i_p and i_q have the same sign in $\boldsymbol{\mathcal{C}}^*$ if and only if $i_q - i_p - (q-p)$ is even. Hence the proposition follows. □

Shephard [36] gives an extension of Gale's criterion to faces of any dimension. Fix a subset W of the linear ordered set $[n]$. A subset $X \subset W$ will be called a *contiguous subset* of W if there is a pair $\{i, j\}$, $1 < i \leq j < n$, such that $X = \{i < i+1 < \cdots < j-1 < j\}$, and $i-1, j+1 \notin W$. X is said to be *even* [resp. *odd*] when $|X|$ is even [resp. odd].

Corollary 4.3 (Gale's evenness criterion for faces, [36]). *Consider the alternating matroid polytope $\mathbb{A}_r([n])$, and suppose $n > r > 2$. Then a k-element subset W of $[n]$, $1 \le k \le r-1$, is a face of rank k of $\mathbb{A}_r([n])$, if and only if W admits at most $r-1-k$ odd contiguous subsets.*

Proof. The result easily follows from Gale's criterion and from the next simple lemma whose proof is left to the reader.

Lemma 4.4. *Let W be a k-element subset of $[n]$. Let m be the number of odd contiguous subsets of W. Then there is a $(k+m)$-element subset F of $[n]$ containing W and such that every contiguous subset of X is even.* $\square$

$\square$

An immediate consequence of Corollary 4.3 is:

Corollary 4.5 ([28]). *Any cyclic matroid polytope is simplicial and neighborly.* $\square$

Remark that Corollary 4.5 is also a simple consequence of Corollary 3.6 and Theorem 3.2.

Let us now examine the role of vertices in the face lattice of cyclic [matroid] polytopes. Suppose that $\boldsymbol{H}$ is an affine hyperplane of $\mathbb{R}^d$ separating strictly the vertex $v \in \mathrm{vert}(\boldsymbol{\mathcal{P}})$ of the other vertices of $\boldsymbol{\mathcal{P}}$. The face lattice of the polytope $\boldsymbol{H} \cap \boldsymbol{\mathcal{P}}$ does not depend on the choice of $\boldsymbol{H}$, and is called the *combinatorial vertex-figure* of $\boldsymbol{\mathcal{P}}$ at v. The combinatorial vertex-figure $\boldsymbol{\mathcal{P}}$ at v is isomorphic to the interval $[v, \boldsymbol{\mathcal{P}}]$ in the face lattice $\mathfrak{Fac}(\boldsymbol{\mathcal{P}})$. It is not difficult to see that the combinatorial vertex-figure of $\boldsymbol{\mathcal{P}}$ at v is isomorphic to the face lattice $\mathfrak{Fac}(\mathbb{A}_{\mathit{ff}}(V)/v)$, where V is the vertex set of the polytope $\boldsymbol{\mathcal{P}}$. More generally we say that the *combinatorial vertex-figure* of the matroid polytope $\boldsymbol{\mathcal{M}}$ at $s \in S(\mathcal{M})$ is the face lattice $\mathfrak{Fac}(\boldsymbol{\mathcal{M}}/s)$ (or equivalently is isomorphic to the interval $[s, S]$ in the face lattice $\mathfrak{Fac}(\boldsymbol{\mathcal{M}})$).

Proposition 4.6. *If r is even [resp. odd] the combinatorial vertex-figure of $\boldsymbol{\mathcal{C}m}_r([n])$ at i, $i \in \{1, n\}$ [resp. $i \in [n]$], is isomorphic to the [face] lattice $\mathfrak{Fac}(\mathbb{A}_{r-1}([n-1]))$.*

Proof. It suffices to consider the case $\boldsymbol{\mathcal{C}m}_r([n]) = \mathbb{A}_r([n])$. It is clear that the acyclic oriented matroid ${}_{\overline{[i-1]}}\mathbb{A}_{r-1}(\{1 < \cdots < \widehat{i} < \cdots < n\})$ coincides with $\mathbb{A}_{r-1}(\{i+1 \prec \cdots \prec n \prec 1 \prec \cdots \prec i-1\})$. Thus the proposition is a consequence of Corollary 3.9. $\square$

Proposition 4.6 shows the strong regularity of cyclic matroid polytopes of odd rank. The special role of the vertices in the case of even rank will be illustrated by the construction of cyclic matroid polytopes of even rank proposed in the next section.

5 Cyclic matroid polytopes: New results

We say that matroid polytope $\boldsymbol{\mathcal{M}}$ is *rigid* if its face lattice $\mathfrak{Fac}(\boldsymbol{\mathcal{M}})$ determines $\boldsymbol{\mathcal{M}}$.

Theorem 5.1 (Cyclic matroid polytopes of odd rank). *The cyclic matroid polytopes of odd rank are rigid. More precisely the geometrical type of* $\boldsymbol{\mathcal{C}m}_{2k+1}([n])$, $k \geq 1$, *is* $\mathbb{A}_{2k+1}([n])$.

Note that Theorem 5.1 is a particular case of the following remarkable result of Ido Shemer [34] (see [3, Theorem 9.4.13] for a matroidal proof):

Theorem 5.2 ([34]). *Every neighborly rank* $2k+1$ *[matroid] polytope is rigid.* $\square$

We present here a short direct proof of Theorem 5.1.

Proof. The case $n = 2k+1$ is trivial. We suppose $n > 2k+1 \geq 3$. We know that $\boldsymbol{\mathcal{C}m}_{2k+1}([n])$ is neighborly, see Corollary 4.5 above. From Proposition 4.1 we conclude that $\boldsymbol{\mathcal{C}m}_{2k+1}([n])$ is a uniform matroid and $|\boldsymbol{\mathcal{C}}^+| = |\boldsymbol{\mathcal{C}}^-| = k+1$ for every signed circuit $\boldsymbol{\mathcal{C}} \in \mathfrak{C}(\mathbb{A}_{2k+1}([n]))$. Let $\mathfrak{F}$ be the set of facets of $\boldsymbol{\mathcal{C}m}_{2k+1}([n])$. The Gale's criterion can be stated as follows (see Corollary 4.3):

- $F \in \mathfrak{F}$ *if and only if every contiguous subset* X *of* F *is even.*

Suppose there are consecutive elements i and $i+1$ contained in a positive part of a signed circuit $\boldsymbol{\mathcal{C}}^+$. Then the number of odd contiguous subsets of $\boldsymbol{\mathcal{C}}^+$ is at most $|\boldsymbol{\mathcal{C}}^+|-2 = k-1$. From Corollary 4.3 we conclude that $\boldsymbol{\mathcal{C}}^+$ is a face of $\boldsymbol{\mathcal{C}m}_{2k+1}([n])$, a contradiction with Corollary 3.4. Thus, $\mathrm{sg}_{\boldsymbol{\mathcal{C}}}(i) = -\mathrm{sg}_{\boldsymbol{\mathcal{C}}}(i+1)$ for every $i \in [n-1]$ and every signed circuit $\boldsymbol{\mathcal{C}}$, $\boldsymbol{\mathcal{C}} \in \mathfrak{C}(\boldsymbol{\mathcal{C}m}_{2k+1}([n]))$, such that $i, i+1 \in \underline{\boldsymbol{\mathcal{C}}}$. Then the equality $\boldsymbol{\mathcal{C}m}_{2k+1}([n]) = \mathbb{A}_{2k+1}([n])$ follows from the Signature Lemma 3.3. $\square$

We remark that for even rank, only a partial structure theorem holds. In particular there are non-realizable rank $2k$ cyclic matroid polytopes. Two elements s, s', $s \neq s'$, of the ground set S of an oriented matroid $\mathcal{M}$ are said to form a *sign-invariant pair*, or shortly an *invariant pair*, if s and s' have always the same sign or always the opposite sign in all the signed circuits of $\mathcal{M}$ containing them. In the first case the pair $\{s, s'\}$ is called *covariant*, in the latter *contravariant*. Note that a covariant [resp. contravariant] pair of $\mathcal{M}$ is a contravariant [resp. covariant] pair of $\mathcal{M}^*$. The *inseparability graph* of $\mathcal{M}$ [32] (*invariance graph* in [11]) is the graph, $IG(\mathcal{M})$, on the vertex set S whose edges are all the invariant pairs of $\mathcal{M}$. Observe that the inseparability graph $IG(\mathcal{M})$ is invariant under orthogonality and reorientations: $IG(\mathcal{M}) = IG(\mathcal{M}^*) = IG({}_{\overline{X}}\mathcal{M})$ for all $X \subset S$.

Remark 5.3. Let $\boldsymbol{\mathcal{P}}$ be a neighborly polytope of dimension $2k \geq 4$. Let V be its vertex set. Then the following two notions are equivalent (we leave the proof to the reader):

- $\{s, s'\}$ is contravariant pair of the affine matroid polytope $\mathbb{A}_{\boldsymbol{ff}}(V)$.
- Set $\text{conv}(s, s') = \ell$. Then $\boldsymbol{\mathcal{P}}/\ell$ is a neighborly polytope of dimension $2k-2$ on a vertex set V' with $|V|-2$ elements: i.e., ℓ is an *universal edge* of the polytope $\boldsymbol{\mathcal{P}}$, in the notations of Ido Shemer [34].

Theorem 5.4 (Cyclic matroid polytopes of even rank). *A rank $2k$, $k \geq 2$, matroid polytope $\boldsymbol{\mathcal{M}}$, on a ground set $S(\boldsymbol{\mathcal{M}})$ with n, $n \geq 2k+2$, elements is a cyclic matroid polytope if and only if there is a covariant pair $\{s, s'\}$ of $\boldsymbol{\mathcal{M}}$ such that both the contractions $\boldsymbol{\mathcal{M}}/s$ and $\boldsymbol{\mathcal{M}}/s'$ are cyclic matroid polytopes.*

Proof. Let us begin with the "only if" part: let $\boldsymbol{\mathcal{M}} = (S, \mathfrak{C})$ be a rank $2k$ cyclic matroid polytope with admissible ordering $S_{\prec} := \{s = s_1 \prec \cdots \prec s_n = s'\}$. Then both $\boldsymbol{\mathcal{M}}/s$ and $\boldsymbol{\mathcal{M}}/s'$ are rank $2k-1$ cyclic matroid polytopes by Proposition 4.6. To obtain a contradiction suppose that $\{s, s'\}$ is not a covariant pair: i.e., there is a signed circuit $\boldsymbol{\mathcal{C}} \in \mathfrak{C}$ such that $s_1 \in \boldsymbol{\mathcal{C}}^+$, and $s_n \in \boldsymbol{\mathcal{C}}^-$. As $|\boldsymbol{\mathcal{C}}| = |\boldsymbol{\mathcal{C}}^+| + |\boldsymbol{\mathcal{C}}^-| \leq 2k+1$, we have necessarily $|\boldsymbol{\mathcal{C}}^+| \leq k$ or $|\mathrm{C}^-| \leq k$. If we have $|\boldsymbol{\mathcal{C}}^+| \leq k$ [resp. $|\boldsymbol{\mathcal{C}}^-| \leq k$], then the set $\boldsymbol{\mathcal{C}}^+ \setminus \{s_1\}$ [resp. $\boldsymbol{\mathcal{C}}^- \setminus \{s_n\}$], of size $k-1$ would be a face of the rank $2k-1$ neighborly matroid polytope $\boldsymbol{\mathcal{M}}/s$ [resp. $\boldsymbol{\mathcal{M}}/s'$]; consequently $\boldsymbol{\mathcal{C}}^+$ [resp. $\boldsymbol{\mathcal{C}}^-$] would also be a face of the simplicial matroid polytope $\boldsymbol{\mathcal{M}}$, a contradiction with Corollary 3.4. We conclude that $\{s, s'\}$ is a covariant pair.

We will prove the "if part" of the theorem. Let H be a facet of $\boldsymbol{\mathcal{M}}$. As $\{s, s'\}$ is a contravariant pair of the dual oriented matroid $\boldsymbol{\mathcal{M}}^*$ we have $H \cap \{s, s'\} \neq \emptyset$. Observing that both the matroids $\boldsymbol{\mathcal{M}}/s$ and $\boldsymbol{\mathcal{M}}/s'$ are uniform (see Theorem 5.1) it follows that $\text{rank}(H) = |H|$, and hence $\boldsymbol{\mathcal{M}}$ is a simplicial matroid polytope. Any rank $2k-2$ face of $\boldsymbol{\mathcal{M}}$ is contained in exactly two facets. Therefore the substitution $s \leftrightarrow s'$ maps a facet to a facet, so $\mathfrak{Fac}(\boldsymbol{\mathcal{M}}/s) \cong \mathfrak{Fac}(\boldsymbol{\mathcal{M}}/s')$. Let us denote by $\overline{s_i}$, $i \in [n-1]$, the vertex of $\boldsymbol{\mathcal{M}}/s$ corresponding to the line $\overline{ss_i}$ joining s and s_i. Every cyclic permutation of an admissible ordering of odd rank cyclic matroid polytope is also an admissible order, by the Gale's evenness criterion. Then there is an admissible order $\overline{S}_{\overline{\prec}} = \{\overline{s}_{i_1} \overline{\prec} \cdots \overline{\prec} \overline{s}_{i_{n-1}} = \overline{s'}\}$ of the vertices of the rank $2k-1$ cyclic matroid polytope $\boldsymbol{\mathcal{M}}/s$. The reader can easily check that the facets of $\boldsymbol{\mathcal{M}}$ satisfy the Gale's criterion relatively to the ordering $S_\prec = \{s \prec s_{i_1} \prec \cdots \prec s_{i_{n-2}} \prec s'\}$. So $\boldsymbol{\mathcal{M}}$ is a cyclic matroid polytope, as required. $\square$

Corollary 5.5. *The contraction $\boldsymbol{\mathcal{C}m}_{2k}([n])/i$, $i \in [n]$, is a cyclic matroid polytope if and only if $i \in \{1, n\}$. Furthermore the cyclic matroid polytope $\boldsymbol{\mathcal{C}m}_{2k}([n])$, $n \geq 2k+2 \geq 4$, has a unique covariant pair $\{1, n\}$. This elements are necessarily the extrema of any admissible ordering.*

Proof. We claim that if $S_\prec = \{j_1 \prec \cdots \prec j_n\}$ is an admissible ordering for $\boldsymbol{\mathcal{C}m}_{2k}([n])$ then $\boldsymbol{\mathcal{C}m}_{2k}([n])/i$, $i \in [n]$, is a cyclic matroid polytope if and only $i \in \{j_1, j_n\}$. By hypothesis $\boldsymbol{\mathcal{C}m}_{2k}([n])$ is combinatorially equivalent to $\mathbb{A}_{2k}(S_\prec)$, $i \mapsto j_i$. From Equality 3.9.2 we known that:

- $\boldsymbol{\mathcal{C}m}_{2k}([n])/1$ is combinatorial equivalent to the alternating oriented matroid $\mathbb{A}_{2k}(S_\prec)/j_1 = \mathbb{A}_{2k-1}(\{\widehat{j_1} \prec j_2 \prec \cdots \prec \cdots \prec j_n\})$.
- For every i, $2 \leq i \leq n$, $\boldsymbol{\mathcal{C}m}_{2k}([n])/i$ is combinatorially equivalent to

$$\mathbb{A}_{2k}(S_\prec)/j_i = {}_{\overline{\{j_1, \cdots, j_{i-1}\}}}\mathbb{A}_{2k-1}(\{j_1 \prec \cdots \prec \widehat{j_i} \prec \cdots \prec j_n\}).$$

By Theorem 5.1 ${}_{\overline{\{j_1, \cdots, j_{i-1}\}}}\mathbb{A}_{2k-1}(\{j_1 \prec \cdots \prec \widehat{j_i} \prec \cdots \prec j_n\})$, where $2 \leq i \leq n$, is a cyclic matroid polytope if and only if is an alternating oriented matroid, hence if and only if $i = n$. Then Corollary 5.5 is a consequence of Theorem 5.4. $\square$

Proposition 5.6. *A rank $2k$, $k \geq 2$, simplicial matroid polytope $\boldsymbol{\mathcal{M}}$ is a cyclic matroid polytope if and only if it has a covariant pair $\{s, s'\}$ such that the contraction $\boldsymbol{\mathcal{M}}/s$ is a cyclic matroid polytope.*

Proof. The proof is similar to that of Theorem 5.4 and left to the reader. $\square$

Admissible orderings of cyclic matroid polytopes $\boldsymbol{\mathcal{C}m}_r([n])$ are characterized in the next theorem. The cases $n = r, r+1$ are trivial and omitted. As the matroid polytopes $\boldsymbol{\mathcal{C}m}_r([n])$ and $\mathbb{A}_r([n])$ are combinatorially equivalent we can reduce our study to the admissible orderings of alternating oriented matroid polytopes.

We define a graph $\mathfrak{G}(n\,;r)$, called the *Gale graph* of $\mathbb{A}_r([n])$, whose vertex set is $[n]$, and such that the pair $\{i,j\} \in [n]\times[n]$ is an edge if and only if the elements i and j are consecutive for some admissible ordering of $\mathbb{A}_r([n])$. We say that the hamiltonian path $[i_1, i_2, \dots, i_n]$ of the Gale graph $\mathfrak{G}(n\,;r)$ *determines* the linear ordering $\{i_1 \prec i_2 \prec \cdots \prec i_n\}$.

Theorem 5.7 (Admissible orderings of cyclic matroid polytopes).

(5.7.1) *The Gale graph* $\mathfrak{G}(n\,;2k{+}1)$, $n > 2k{+}2 > 3$, *is the hamiltonian cycle* $[1, \dots, n, 1]$. $\mathbb{A}_{2k+1}([n])$ *has* $2n$ *admissible orderings determined by the* $2n$ *hamiltonian paths of* $\mathfrak{G}(n\,;2k+1)$.

(5.7.2) *The Gale graph* $\mathfrak{G}(n\,;2k)$, $n > 2k+1 > 3$, *is the union of the two cycles* $[2,3,\dots,n,2]$ *and* $[1,2,\dots,n{-}1,1]$. $\mathbb{A}_{2k}([n])$ *has* 4 *admissible orderings determined by the* 4 *hamiltonian paths of* $\mathfrak{G}(n\,;2k)$ *whose extrema are the elements* 1 *and* n.

Proof. A facet F of $\mathfrak{Fac}(\mathbb{A}_{2k+1}([n]))$ is said to be a *special facet* when there are exactly two elements $h_1, h_2 \in [n] \setminus F$ such that, for every $f \in F$, either $F \setminus \{f\} \cup \{h_1\}$ or $F \setminus \{f\} \cup \{h_2\}$ is again a facet of $\mathfrak{Fac}(\mathbb{A}_{2k+1}([n]))$. It is straightforward to prove that there are exactly n special facets: $F_i := \{i, \dots, i{+}2k\}$, $\pmod n$, $i \in [n]$. The set of special facets of $\mathfrak{Fac}(\mathbb{A}_{2k+1}([n]))$ does not depend on the admissible ordering of $\mathbb{A}_{2k+1}([n])$. Then the unique admissible orderings of $\mathbb{A}_{2k+1}([n])$ are of type $\{\omega(1) \prec_\omega \cdots \prec_\omega \omega(n)\}$ where ω denotes a circular permutations of $\{1,2,\dots,n\}$. We conclude also that the Gale graph $\mathfrak{G}(n\,;2k+1)$ is the hamiltonian cycle $[1,2,\dots,n,1]$.

From Corollary 5.5 we know that the elements 1 and n are the extrema of any admissible ordering of $\mathbb{A}_{2k}([n])$ and the contractions $\mathbb{A}_{2k}([n])/1$ and $\mathbb{A}_{2k}([n])/n$ are cyclic matroid polytopes. As $n > 2k{+}1 \geq 5$, we have $(n{-}1) > 2(k{-}1){+}1 \geq 3$. Making use of the case (5.6.1) relatively to the cyclic matroid polytope $\mathbb{A}_{2k}([n])/1$ we know that the hamiltonian paths of the cycle [2,3, ... ,n,2] determine the admissible orderings of $\mathbb{A}_{2k}([n])/1$. From these data and making use of Gale's criterion, we conclude that $\{i_1 \prec \cdots \prec i_{n-1} = n\}$ [resp. $\{n = i_1 \prec \cdots \prec i_{n-1}\}$] is an admissible ordering of $\mathbb{A}_{2k}([n])/1$ if and only if $\{1 \prec i_1 \prec \cdots \prec i_{n-1} = n\}$ [resp. $\{n = i_1 \prec \cdots \prec i_{n-1} \prec 1\}$] is an admissible ordering of $\mathbb{A}_{2k}([n])$. A similar result holds to $\mathbb{A}_{2k}([n])/n$. We conclude that the Gale graph $\mathfrak{G}(n\,;2k)$ is the union of the two cycles $[1,2,\dots,n-1,1]$ and $[2,3,\dots,n,2]$. □

From Theorem 5.4 it is possible to construct the geometrical types of all cyclic polytopes with even rank (see Remark 5.12 below for the realizable case). However it is not clear, at this stage, whether there are non-realizable cyclic matroids of even rank. The following result of Richter & Sturmfels settles the question.

Theorem 5.8 ([31]). *There is a non-realizable rank* 4 *cyclic [uniform] matroid polytope with* 10 *vertices. A rank* $2k$ *cyclic matroid polytope is realizable if every minor by deletion is cyclic.* □

We give two results that emphasize the very special place of alternating oriented matroids among realizable cyclic matroid polytopes of even rank.

Theorem 5.9. *Let* $\boldsymbol{\mathcal{M}}$ *be a matroid polytope of even rank* $2k$, $k \geq 2$, *on a ground set* S *with* n, $n \geq 2k+2$, *elements. Then the following properties are equivalent:*

(5.9.1) $\boldsymbol{\mathcal{M}} = \mathbb{A}_{2k}(S_{\prec})$ *for some linear ordering* $S_{\prec}$ *of the elements of* S.

(5.9.2) $\boldsymbol{\mathcal{M}}$ *is a cyclic uniform matroid polytope and its inseparability graph is a hamiltonian cycle.*

(5.9.3) *For some element* $s \in S$, *both matroids* $\boldsymbol{\mathcal{M}} \setminus s$ *and* $\boldsymbol{\mathcal{M}}/s$ *are alternating oriented matroid with respect to the same linear ordering of* $S \setminus \{s\}$.

We make use of the following Proposition.

Proposition 5.10 ([11]). *Let* $\mathbb{U}_r = (S, \mathfrak{C})$ *be a rank* r *oriented uniform matroid on a ground set* S *with* n, $n \geq r+2 \geq 4$, *elements. The inseparability graph* $IG(\mathbb{U}_r)$ *is either a hamiltonian cycle or a disjoint union of* $k, k \geq 2$, *paths.* $IG(\mathbb{U}_r)$ *is a hamiltonian cycle if and only if there is a linear ordering of the elements of* S, *say* $S_{\prec} := \{s_1 \prec \cdots \prec s_n\}$, *such that* $\mathbb{U}_r$ *is a reorientation of* $\mathbb{A}_r(S_{\prec})$. *Otherwise* $\mathbb{A}_r(S_{\prec})$, $n \geq r+2 \geq 4$, *is the only oriented uniform matroid whose inseparability graph possesses a hamiltonian path* $[s_1, s_2, \ldots, s_n]$ *where all edges are contravariant.* □

Proof. Implications (5.9.1) $\Longrightarrow$ (5.9.2), (5.9.3) are trivial.
We will prove (5.9.2) $\Longrightarrow$ (5.9.1). From Corollary 5.5 we know that $\boldsymbol{\mathcal{M}}$ has exactly one covariant pair $\{s, s'\}$ and these elements are the extrema of any admissible ordering of $\boldsymbol{\mathcal{M}}$. Suppose that $[s_1 = s, s_2, \ldots, s_n = s', s]$ is a hamiltonian cycle of the inseparability graph of $\boldsymbol{\mathcal{M}}$. We know that the pairs $\{s_i, s_{i+1}\}$, $i \in [n-1]$, are necessarily contravariant. Then the inseparability graph $IG(\boldsymbol{\mathcal{M}})$ possesses the hamiltonian path $s_1, s_1, \ldots, s_n$ where all edges are contravariant and the implication follows from Proposition 5.10.

We will prove (5.9.3) $\Longrightarrow$ (5.9.2). Since both $\mathcal{M}/s$ and $\mathcal{M} \setminus s$ are uniform matroids, $\mathcal{M}$ is also uniform. Let $\{s_1 \prec \cdots \prec s_{n-1}\}$ be an ordering of $S \setminus \{s\}$, with respect to which both $\mathcal{M}/s$ and $\mathcal{M} \setminus s$ are alternating oriented matroids. Then, for every i, $i \in [n-2]$, the pair $\{s_i, s_{i+1}\}$ is a contravariant pair of $\mathcal{M}$. We claim that the existence of a chain of length $n-2$ in $IG(\mathcal{M})$ implies that this graph is a hamiltonian cycle. Indeed the element s can not be an isolated vertex from a result of Roudneff [32] (see also [3, Theorem 7.8.6]). From Proposition 5.10 we conclude that $IG(\mathcal{M})$ is a hamiltonian cycle. To finish the proof it is enough to see that one of the pairs $\{s, s_1\}$ or $\{s, s_{n-1}\}$ is contravariant and the other covariant. Suppose for a contradiction that both the pairs are covariant or contravariant. Then $\{s_1, s_{n-1}\}$ is a covariant pair of both the alternating oriented matroid $\mathcal{M} \setminus s$ and $\mathcal{M}/s$. From Equation (3.3.1) we know that $\mathrm{sg}_{\mathcal{C}}(s_1) = (-1)^r \mathrm{sg}_{\mathcal{C}}(s_{n-1})$, [resp. $\mathrm{sg}_{\mathcal{C}'}(s_1) = (-1)^{(r-1)} \mathrm{sg}_{\mathcal{C}'}(s_{n-1})$] for every signed circuit $\mathcal{C}$ of $\mathcal{M} \setminus s$, [resp. $\mathcal{C}'$ of $\mathcal{M}/s$] such that $s_1, s_{n-1} \in \underline{\mathcal{C}}$ [resp. $s_1, s_{n-1} \in \underline{\mathcal{C}'}$], a contradiction. $\square$

Proposition 5.11. *For any pair of integers $\{n, k\}$, $n \geq 2k+3 \geq 7$, there is a rank $2k$ matroid polytope $\mathcal{M}$ with n elements and the following properties:*

- *$\mathcal{M}$ is a realizable cyclic uniform matroid polytope but not an alternating oriented matroid.*
- *All proper submatroids of $\mathcal{M}$ are cyclic uniform matroid polytopes.*

Proof. Consider the alternating oriented matroid $\mathbb{A}_{2k}([n-1])$, where $n-1 \geq 2k+2 \geq 6$. Consider the basis $B = \{1, 2, n-2(k-1), n-2k+1, n-1\}$ of $\mathbb{A}_{2k}([n-1])$. Set $B_{\prec} := \{1 \prec 2 \prec n-1 \prec n-2 \prec n-3 \prec \cdots \prec n-2(k-1)\}$. Let $\mathcal{M}$ be the [realizable] rank $2k$ uniform oriented matroid on the ground set $[n]$ determined by the following rules:

- $\mathcal{M}$ is a single element extension of $\mathbb{A}_{2k}([n-1])$.
- Let $\mathfrak{C}^*_n$ be the set of signed cocircuits of $\mathcal{M}$ containing the element n. If $\mathcal{C}^* \in \mathfrak{C}^*_n$, then $((\mathcal{C}^*)^+ \setminus \{n\}; (\mathcal{C}^*)^- \setminus \{n\}))$ is a signed cocircuit of $\mathbb{A}_{2k}([n-1])$.
- $\mathrm{sg}_{\mathcal{C}^*}(n) = (-1)^j \ \mathrm{sg}_{\mathcal{C}^*}(i_j)$ if i_j is the smallest element of $B_{\prec}$ that occurs in $\underline{\mathcal{C}}^*$ and the j-th element of $B_{\prec}$.

It is clear that there is a positive signed cocircuit of $\mathcal{M}$ supported by $\{2, 3, 2k+2, \ldots, n\}$. Then $\mathcal{M}$ is an acyclic oriented uniform matroid. Proposition 5.11 results of the following statements (the straightforward proofs are left to the reader):

(5.11.1) $\mathcal{M} \setminus 1 = \mathbb{A}_{2k}(\{\widehat{1} < 2 < \cdots < n\})$.

(5.11.2) $\mathcal{M} \setminus n = \mathbb{A}_{2k}([n-1])$.

(5.11.3) $\mathcal{M}/1 = \mathbb{A}_{2k}(\{3 \prec 4 \prec \cdots \prec n-1 \prec 2 \prec n\})$.

(5.11.4) $\mathcal{M}/n = \mathbb{A}_{2k}(\{1 \prec 3 \prec 4 \prec \cdots \prec n-1 \prec 2)\}$.

(5.11.5) $\mathcal{M}$ is not an alternating oriented matroid. □

Remark 5.12 (Realizable cyclic matroid polytopes of rank 2k). Assuming $n \geq 2k+3 \geq 5$, we choose a $2k$ dimensional cyclic polytope $\mathcal{P}_{\rm o}$ in $\mathbb{R}^{2k}$ with vertex set $V = \{v_2, v_3, \ldots, v_{n-1}\}$ and admissible ordering $V_{\prec} := \{v_2 \prec \cdots \prec v_{n-1}\}$. We choose the origin O in the exterior of $\mathcal{P}_{\rm o}$ but very close to a facet $F_{\rm o}$ of $\mathcal{P}_{\rm o}$ so that the hyperplanes supporting others facets of $\mathcal{P}_{\rm o}$ do not separate O from $\mathcal{P}_{\rm o}$. Now, identifying $\mathbb{R}^{2k+1}$ to $\mathbb{R}^{2k} \times \mathbb{R}$, we set $v_1 = (0, -1)$ and $v_n = (0, 1)$. Consider the polytope $\mathcal{P}_1 := \text{conv}(v_1, \ldots, v_n)$. It is clear that

$$\mathbb{A}_{\mathit{ff}}(\{v_1, \ldots, v_n\}) = \mathbb{A}_{2k+2}(\{v_1 \prec \cdots \prec v_n\}).$$

Since $\mathcal{P}_1$ is simplicial, small perturbations of its vertices keep its combinatorial type: if $w_2, \ldots, w_{n-1}$ are points chosen in general position in small balls centered at $v_2, \ldots, v_{n-1}$, respectively, then

$$\mathcal{P} := \text{conv}(v_1, w_2, w_3, \ldots, w_{n-1}, v_n)$$

is a cyclic $2k+1$-polytope. The reader may check that this process produces any realizable cyclic polytope of rank $2k$, $k \geq 2$.

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